

Asymptotic Bounds on Minimum Number of Disks Required to Hide a Disk

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Abstract—We consider the problem of blocking all rays emanating from a closed unit disk with a minimum number of closed unit disks in the two-dimensional space, where the minimum distance from a disk to any other disk is given. We study the asymptotic behavior of the minimum number of disks as the minimum mutual distance approaches infinity. Using a regular ordering of disks on concentric circular rings we derive an upper bound and prove that the minimum number of disks required for blocking is quadratic in the minimum distance between the disks.

Keywords—asymptotic bounds; blocking set; hiding disk.

I. INTRODUCTION

Let U be a closed unit disk, i.e., a disk with radius 1, in the two-dimensional plane and let \mathcal{R} denote the set of all rays that emanate from U . A ray $r \in \mathcal{R}$ is said to be blocked by a disk δ if r and δ have a non-empty intersection. A set \mathcal{D} of closed unit disks, with $U \notin \mathcal{D}$, is called a *blocking set* if every ray $r \in \mathcal{R}$ is blocked by a disk in \mathcal{D} . In addition, a blocking set \mathcal{D} is called *d -apart* if the distance between each pair of disks in $\mathcal{D} \cup \{U\}$ is at least d , where distances are measured from center to center.

Minimum Cardinality Blocking Set Problem. *Given d , what is the minimum cardinality N_d of a d -apart blocking set?*

More specifically, we are interested in the asymptotic behavior of N_d , as d tends to infinity. For reasons of convenience, we focus on the following problem, which is equivalent to the minimum cardinality blocking set problem.

Maximum Distance Blocking Set Problem. *Given N unit disks, what is the maximum distance d for which the disks may form a d -apart blocking set?*

Motivation. The problems considered in this paper are related to occlusion problems in table-top interaction devices, where multiple sensors, for example, light sensors or cameras, scan the two-dimensional plane just above the table's surface for objects like game pieces or fingers. A circular object emitting light in that plane cannot be "seen" by the sensors, in other words, it is no longer visible if all rays emanating from it are blocked by other circular

objects, for example. The results presented in this paper give a valuable insight on the number of objects required for one such occlusion problem to occur. In other words, we explored one of the "limitations" of the described technology for object detection and by presenting the results, we showed that the occlusion problems can be easily avoided in practice, using a small number of objects, for instance, or designing an application in such a way, that it does not allow objects to be relatively close one to another.

Our Contributions. In this paper we show that both upper and lower bounds on the minimum number N_d of disks are quadratic in d , i.e., we prove that $N_d = \Theta(d^2)$. In more detail, we first show that $N \geq 6$ disks can be positioned such that they form a 2-apart blocking set. The disks of that blocking set are placed on a circle concentric to U with neighboring disks being mutually tangent. We present a simple algorithm of pushing the disks towards the center of U such that the blocking of rays is preserved. The algorithm provides a regular ordering of disks on concentric circular rings such that the disks form a d -apart blocking set, where $d > 2$. This is used to show that

$$\frac{\pi^2}{16} \leq \lim_{d \rightarrow \infty} \frac{N_d}{d^2} \leq \frac{\pi^2}{2},$$

where the lower bound is derived as an immediate consequence of the existing lower bound in [7].

Related Work. Jovanović, Korst and Janssen [6] consider a variant of the above blocking set problem, where they consider blocking all lines intersecting a given unit disk, instead of blocking all rays emanating from a given unit disk. The authors presented upper and lower bounds for small values of the minimum mutual distance d between the disks, namely, for $2 \leq d \leq 4$. Jovanović et al. [7] show that the minimum number of unit disks needed to block all rays emanating from a single point is quadratic in d . In addition, we refer to Fulek, Holmsen and Pach [3], who focus on hitting a maximum number of disks with one ray from an arbitrary point, while we aim at blocking all rays emanating from a given disk with a minimum number of disks. The problem of our interest is also related to the

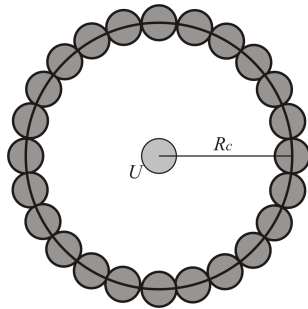


Figure 1. 24 disks positioned on the circle of radius R_c concentric to U .

work of Dumitrescu and Jiang [1] and Mitchell [2], where the authors consider an illumination problem for maximal disk packings by proving the existence of points that are not visible from outside a disk packing. We are not aware of other work that is closely related, although there are many more remotely related visibility problems; see e.g. Chapter 28 on visibility by O'Rourke in [8] or the work presented in [9], [10], [11]. For further details on object detection on related table-top devices we refer to [4], [5].

Overview. The rest of the paper is organized as follows. In Section II we present a construction of a 2-apart blocking set and a method that transforms the constructed blocking set into a d -apart blocking set. In Section III we introduce an ordering of disks on circular rings with which we maximize the distance d between the disks of the blocking set, and we present a simple algorithm that for a given number of disks determines the described ordering. Section IV gives upper and lower bounds on the minimum number of disks required to hide a disk. We conclude the paper with the discussion in Section V.

II. BLOCKING RAYS

In this section we propose an ordering of disks that enables blocking all rays from \mathcal{R} for a given number N of disks. We assume for convenience that $N = 6n$. The N disks are placed on a circle c concentric to the given disk U , such that the centers of the disks are on the circle c and there is no gap between neighboring disks; see Figure 1.

More precisely, two neighboring disks positioned on c are mutually tangent. The radius R_c of circle c is easily derived from $R_c = 1/\sin \frac{\pi}{6n}$. Given the mutual tangency of each pair of neighboring disks, one can easily see that any ray $r \in \mathcal{R}$ is blocked by at least one and at most two disks of the given set of $6n$ disks. Hence, these disks form a blocking set. The distance between the neighboring disks on c is 2, while the distance between U and a disk from the blocking set is at least 2 for any $n \geq 1$. Therefore, the constructed blocking set is 2-apart. Let this blocking set be denoted by \mathcal{D}_2 .

For the maximum distance blocking set problem, we are interested in the maximum distance d for which the $6n$ disks

form a d -apart blocking set for \mathcal{R} . As such, the problem appears to be hard: constructing a d -apart blocking set for an arbitrary d is certainly challenging, because it requires proving that a set of N disks is a blocking set. Therefore, we focus on *transforming* the constructed 2-apart blocking set into a d -apart blocking set.

In order to transform \mathcal{D}_2 into a d -apart blocking set, with $d > 2$, the disks of \mathcal{D}_2 should be separated from each other, while the blocking of all rays should be preserved. Let us next describe one step of the proposed transformation.

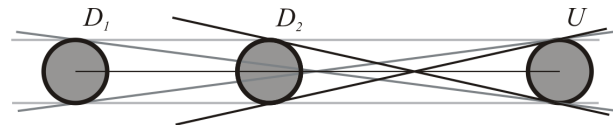


Figure 2. Each ray blocked by D_1 is also blocked by D_2 .

Let D_1 and D_2 be two unit disks such that their centers and the center of the given disk U are collinear and D_2 is between U and D_1 ; see Figure 2. Let \mathcal{R}_1 and \mathcal{R}_2 denote the sets of rays blocked by disks D_1 and D_2 , respectively. Since each ray r that is blocked by D_1 is also blocked by D_2 , as shown in [6], we can conclude that $\mathcal{R}_1 \subset \mathcal{R}_2$.

Hence, *the rays blocked by a given disk D are still blocked by D after the disk is moved towards the center of U , i.e., along the line segment that connects the two disks' centers.* Consequently, a transformation of the blocking set \mathcal{D}_2 where some disks of \mathcal{D}_2 are shifted from their original position on circle c towards the center of U represents a transformation into a d -apart blocking set, where d is the minimum of all pair-wise distances between the disks; see Figure 3. The problem of interest to us now is to determine the maximum d for which we can transform \mathcal{D}_2 into a d -apart blocking set.

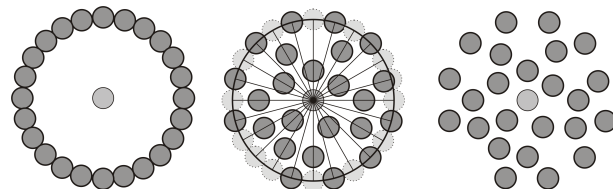


Figure 3. Transformation of \mathcal{D}_2 into a d -apart blocking set.

III. ORDERING DISKS ON CIRCULAR RINGS

In Section II we proved that we can construct blocking sets by pushing the disks of \mathcal{D}_2 into the interior of the circle c , given that the disks are moved in the direction of the center of c . In this section we propose a regular ordering of disks forming a blocking set that can be obtained as follows.

Let \mathcal{D}_2 be the 2-apart blocking set constructed as in Section II, consisting of $6n$ disks. In the interior of the circle c we can define a number of circles called *rings* and

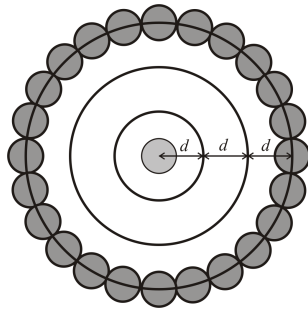


Figure 4. The definition of three circular rings with radii d , $2d$ and $3d$.

denoted as c_1, c_2, \dots, c_k , where the radius of the ring c_1 is d , the radius of c_2 is $2d$, etc. The last ring c_k with the radius kd is assumed to be the given circle, which has radius $R_c = 1/\sin \frac{\pi}{6n}$; see Figure 4. In the process of shifting the disks of \mathcal{D}_2 towards the center, we place the center of each of them exactly on one of the rings.

The line segment that connects the center of a disk in \mathcal{D}_2 and the center of U is called a *thread*. Thus, the disks of \mathcal{D}_2 define $6n$ threads. Since we chose to place the disks on the rings and the disks can be moved only along their threads, each disk can be placed in one of the k intersection points of its thread and the k rings. Note that the d -apart rings ensure that the distance between any two disks positioned on *different rings* is at least d . However, choosing an arbitrary ring for each disk may result in two disks of the *same ring* being less than distance d apart; see Figure 5.

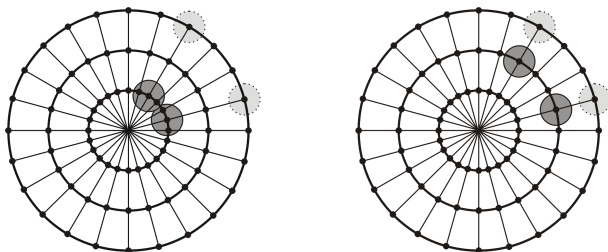


Figure 5. Shifting two disks onto inner rings: left, the disks are not d -apart, and right, the disks are d -apart.

The number k of rings determines the distance d for given n . Given that the radius of the largest ring is $R_c = 1/\sin \frac{\pi}{6n}$ and as we mentioned above $R_c = kd$, we have that

$$d = \frac{1}{k \sin \frac{\pi}{6n}}. \tag{1}$$

Hence, in order to maximize the distance d , we need to minimize the number k of rings needed, for $6n$ disks to form a d -apart blocking set.

For a ring of given radius, it is easy to determine the maximum number of disks that can be positioned equally spaced, such that the distance between two neighboring disks on this ring is at least d . For example, at most 6 disks can

be placed on the first ring, at most 12 disks on the second ring, at most 18 disks on the third ring, etc. In this way, we can easily derive a lower bound on the minimum number k of rings needed, for a given n . However, the minimum number of rings that suffices for disks to form a d -apart blocking set is often larger than this lower bound. This is because of the restriction of fixed positions for placing the disks, which does not always allow placing the maximum number of disks on the rings. In the construction we propose, we place less than maximum disks on some of the rings or even keep some of the rings empty.

In more detail, we choose to place $6n_j$ disks on the j -th ring, where

$$n_j = 2^{\lfloor \log_2 j \rfloor}, \tag{2}$$

such that the disks form a regular polygon. Note that $6n_j$ is *equal* to the maximum number of disks that can be placed, only for the rings $j = 2^l$, for some $l \geq 0$, however, it is *less* than maximum for all other rings; see the comparison given in Table I. For symmetry reasons, we focus on one

Ring j	1	2	3	4	5	6	7	8
Max disks	6	12	18	24	30	36	42	48
$6n_j$	6	12	12	24	24	24	24	48

Table I
THE MAXIMUM NUMBER OF DISKS AND THE CHOSEN NUMBER OF DISKS FOR RINGS 1 TO 8.

of the six sections of \mathcal{D}_2 with n disks. We show that any set of n disks can be split into k subsets, where the j -th subset contains either $2^{\lfloor \log_2 j \rfloor}$ or 0 disks. The j -th subset is then placed on the j -th ring such that the distance between each two disks is at least d . More precisely, we show that the given number n can be represented as

$$n = \bar{n}_1 + \bar{n}_2 + \dots + \bar{n}_k, \tag{3}$$

where $\bar{n}_j \in \{0, n_j\}$, or simplified, any natural number n can be represented as

$$n = b_0 + \underbrace{2 + 2}_{\max 2} + \underbrace{4 + 4 + 4 + 4}_{\max 4} + \dots + \underbrace{2^t + 2^t + \dots + 2^t}_{\max 2^t}, \tag{4}$$

for some $t \geq 0$ and $b_0 \in \{0, 1\}$. Note that the total number of addends in (3) is k , i.e., each addend corresponds to a ring, more precisely, to the number of disks placed on each of the six sections of the ring. This results in including the zero-addends in counting, since they indicate the presence of empty rings. More precisely, we include the zero-addends in counting when we have less than the maximum number of equal addends, for all addends except for the largest ones. For example, $n = 15$ can be represented as $15 = 1 + 2 + 0 + 4 + 4 + 4$ and the number of rings needed is $k = 6$, with the third ring being empty.

Formally, we prove the existence of a representation of n in form (4), using the following lemma.

Lemma 1: For any positive integer n a sequence $A_n = (a_0, a_1, \dots, a_t)$ exists such that

$$n = \sum_{i=0}^t a_i \cdot 2^i \tag{5}$$

where $0 \leq a_i \leq 2^i$ and $a_t > 0$.

Proof: The proof of the lemma follows from the binary scale representation of n . ■

For a given n , there are generally multiple sequences A_n . From Equation (1), to construct a d -apart blocking set, where distance d is as large as possible, we need to minimize the number k of rings. The number of rings we define is equal to the number of addends in (4). Hence, the number k of rings is given by

$$k = (1 + 2 + 4 + 8 + \dots + 2^{t-1}) + a_t = 2^t - 1 + a_t \tag{6}$$

where a_t is the number of addends of size 2^t in (4). Hence, our interest is in the sequences A_n^* for which $2^t + a_t$ is minimal.

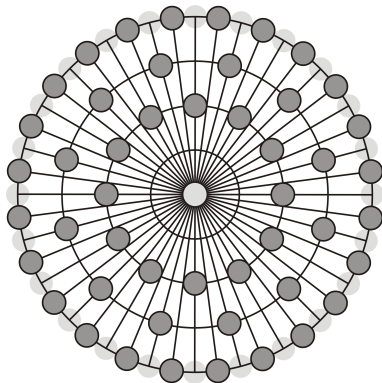


Figure 6. An example of a d -apart blocking set for $n = 8$, where $d \approx 4$.

A. Disk ordering algorithm

In the previous section we showed how to determine the number of rings and the number of disks on each of them, using Lemma 1 and choosing the sequence A_n^* for which the number of rings is minimal. In this section, we present an algorithm that given the sequence A_n^* , for each disk of \mathcal{D}_2 determines the ring on which it should be placed, which results in the disks forming a d -apart blocking set; see Figure 6.

We restrict ourselves to finding the solutions for all n that are divisible by their largest addend 2^t in the representation (4). Note that $2^t | n$ implies that $n_j | n$, for all j .

Let us define a table T with k rows and n columns, such that each thread corresponds to one column of T and each ring corresponds to one row of T , with the outermost ring corresponding to the top row. Each cell of the table T then

represents a position on which the corresponding disk can be placed, i.e., it is the intersection of its thread and a ring. When one disk is moved to a certain position, the value in the corresponding cell of T is set to 1 or “full”, while the other cells of the same column have values 0 or “empty”; see Figure 7. The defined table represents one of the six identical sections of the blocking set, thus, we consider the table as if its columns are cyclic (its first and its last column are connected).

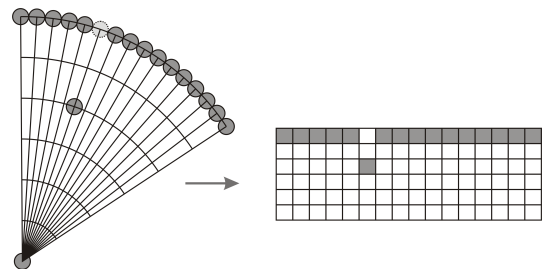


Figure 7. A set of 16 disks with 6 rings and the corresponding 6 x 16 table.

An ordering of full cells in a table T is called *valid* if and only if the following conditions hold:

- There is exactly one full cell in each column;
- The j -th row is either empty or it contains exactly n_j full cells;
- The number of empty cells between any two successive full cells of the j -th row is exactly $\frac{n}{n_j} - 1$.

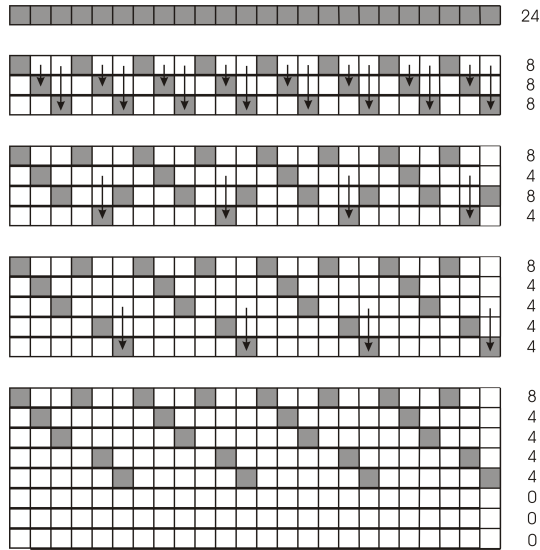
Lemma 2: A valid table T exists for any positive integer n represented by (4) for which $2^t | n$.

Proof: The proof of the lemma is given by a method for constructing a valid table, which follows from the equation $2^m = 2^{m-1} + 2^{m-1}$. In more detail, a complete row of full cells can be split into $n/2^t$ rows, where each row contains 2^t full cells, as illustrated in Figure 8. Each of the resulting rows can again be split into two rows, by pushing every second full cell to a new row. After a finite number of “splitting” steps, each row corresponds to a non-zero addend in representation (4). The rows can be swapped then if necessary, such that each row r that is directly above a row r' contains at least the same number of full cells as r' . The process is completed by inserting empty rows where needed. ■

Note that the proof of Lemma 2 represents a *disk ordering algorithm* that for each of the n disks determines the ring on which it should be placed, such that the disks form a d -apart blocking set.

IV. UPPER AND LOWER BOUNDS

In Sections II and III, we showed that we can construct a d -apart blocking set for each n that is divisible by its largest addend in representation (4). In this section, we present upper and lower bounds on the cardinality N_d of such a


 Figure 8. Constructing a valid table for $n = 24$.

blocking set, as a function of the minimum distance d . We start by deriving an upper bound.

One can easily show that the ordering of disks presented in Section III-A implies that the minimum of all pairwise distances between the disks is d . The relation between the distance d , the given number n and the corresponding number k of rings is given by

$$d = \frac{1}{k \sin \frac{\pi}{6n}} \quad (7)$$

From the choice of sequence A_n^* in Lemma 1, for which $a_t + 2^t$ is minimal, we have that

$$\sum_{j=0}^{t-1} 2^{2j} + (a_t - 1) \cdot 2^t \leq n \quad (8)$$

where a_t is the number of largest addends 2^t in representation (4). From (8) and

$$\sum_{j=0}^{t-1} 2^{2j} = \frac{1}{3}(4^t - 1) \quad (9)$$

it follows that

$$4^t + 3(a_t - 1)2^t \leq 3n + 1 \quad (10)$$

With further transformations of inequality (10) we have

$$\begin{aligned} ((2^t)^2 + 2(a_t - 1)2^t) + (a_t - 1)2^t &\leq 3n + 1 \\ \Leftrightarrow k^2 + (a_t - 1)(2^t - a_t + 1) &\leq 3n + 1 \end{aligned} \quad (11)$$

Since $1 \leq a_t \leq 2^t$, we have that

$$(a_t - 1)(2^t - a_t + 1) \geq 0 \quad (12)$$

Finally, from (11) and (12), we bound the number k of rings by a function in n as follows.

$$k \leq \sqrt{3n + 1} \quad (13)$$

We transform (7) into

$$\frac{1}{kd} \leq \sin \frac{\pi}{6n} \quad (14)$$

and multiply (13) by \sqrt{n}

$$k\sqrt{n} \leq \sqrt{3n^2 + n} \quad (15)$$

Multiplication of (14) and (15) and expressing the limit for $d \rightarrow \infty$, results in

$$\lim_{d \rightarrow \infty} \frac{n}{d^2} \leq \frac{\pi^2}{12} \quad (16)$$

and since $N = 6n$, we derived an upper bound on N_d , i.e.,

$$\lim_{d \rightarrow \infty} \frac{N_d}{d^2} \leq \frac{\pi^2}{2} \quad (17)$$

In [7], the authors proved that the lower bound on the minimum number of disks which form a d -apart blocking set for the set of all rays emanating from a single point is $\frac{\pi^2}{16}d^2$, as d tends to infinity. To block the rays emanating from a given unit disk we need at least as many as to block the rays emanating from its center. Hence, the lower bound on the minimum number N_d of disks is given by

$$\lim_{d \rightarrow \infty} \frac{N_d}{d^2} \geq \frac{\pi^2}{16}. \quad (18)$$

Combining the results of (17) and (18), we proved the following theorem.

Theorem 1: For the minimum cardinality N_d of a d -apart blocking set to block all rays emanating from a unit disk we have

$$\frac{\pi^2}{16} \leq \lim_{d \rightarrow \infty} \frac{N_d}{d^2} \leq \frac{\pi^2}{2}.$$

V. CONCLUSION

We expect that both bounds, especially the upper bound, can be further improved. The following discussion provides some directions for potential improvements.

Constructing a d -apart blocking set from \mathcal{D}_2 through a sequence of transformation steps where a number of disks is pushed towards the center results in the rather large constant $\pi^2/2$. The disks pushed inside circle c block much larger sets of rays than the sets of rays they block from their original positions on c . Consequently, the sets of rays blocked by two disks on different rings may not be disjoint. This implies that constructing blocking sets for which the overlap of sets of blocked rays is minimized may potentially provide a better upper bound. In addition, the number of disks on one ring is less than the maximum possible number for the majority of rings. Placing the maximum number of disks on each of the rings may further improve the upper bound.

The combination of the last two conjectures may be used to define an optimization problem, similar to the problem of opening a combination lock with k rings, i.e., to find the rotation angle for each of the k rings that are d -apart and contain the maximum number of d -apart disks, such that the disks form a blocking set and the total overlap of blocked rays is minimized. We expect that the solution of this problem provides a better upper bound. The main challenge here is still the problem of proving that a set of disks, positioned following some constraints, is a blocking set for the set \mathcal{R} of all rays.

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