Generalised Atanassov Intuitionistic Fuzzy Sets

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***Abstract*—When Atanssov created Intuitionistic Fuzzy Sets, he imposed the condition that the sum of membership and non-membership values for each point in the universe of discourse should be less than or equal to one. We challenge this constraint and define some new types of Intuitionistic Fuzzy Sets such that, for any point in the universe of discourse, the sum of membership and non-membership values can be greater than one, or their difference can be negative or positive, while one value is greater than the other, or the sum of their squares is less than or equal to one.**

***Keywords*-Intuitionistic fuzzy set, Interval-valued fuzzy sets.**

I. INTRODUCTION

`Fuzzy Sets` concept was introduced by Zadeh [1] in 1965. Given an non-empty universe of discourse X, one can define a fuzzy set A based on its membership function $\mu_A : X \to [0,1]$, that is A is a set with 'vague boundary' when compared with crisp sets, where $\mu_A : X \to \{0,1\}$. Of course, the bigger the value of $\mu_A(x)$ is, the greater the degree of membership of x to A is, so $\mu_A(x) = 1$ represents the full membership of x to A.

In 1983, Atanassov generalized the concept of fuzzy set by using two membership functions for the elements of the universe of discourse. The English version appeared in 1986 [2].

Definition 1. *Let X be an non-empty universe of discourse. An* `(Atanassov's) Initutionistic Fuzzy Set` *(AIFS or IFS) is described by:*

$$A = \{(x, \mu_A(x), \nu_A(x)) \,|\, x \in X\} \tag{1}$$

where μ_A is used to define the degree of membership (membership function) and ν_A is used to define the degree of non-membership (non-membership function) of x to A

$$\mu_A : X \to [0,1] \qquad \nu_A : X \to [0,1] \tag{2}$$

satisfying the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1, \; \forall x \in X \tag{3}$$

The word `intuitionistic` was added to suggest that the principle of excluded middle does not hold, so to say $\mu_A(x) + \nu_A(x) = 1$ is not true for all $x \in X$ if one interprets ν as a sort of negation of μ. Some operations on IFSs have been also introduced in [2]:

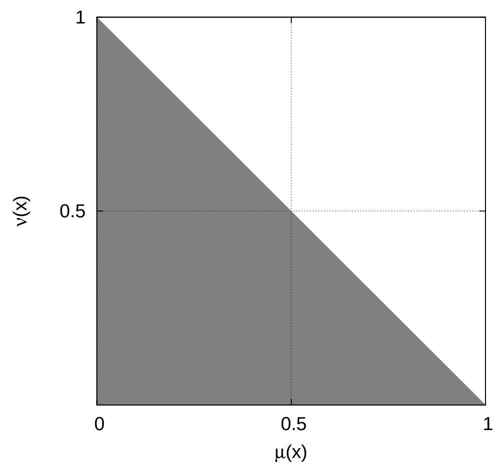

Fig. 1. Atanassov Intuitionistic Fuzzy Set, $\mu_A(x) + \nu_A(x) \le 1, \forall x \in X$

Definition 2. *Given two IFSs A and B over an universe of discourse X, one can define the following relations:*
$A \subset B$ *iff* $\forall x \in X\; \mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x)$
$A = B$ *iff* $A \subset B$ *and* $B \subset A$
as well as the following operations [2]:
$\bar{A} = \{(x, \nu_A(x), \mu_A(x) \,|\, x \in X\}$
$A \cap B = \{(x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)) \,|\, x \in X\}$, *where*
$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$ *and*
$\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$
$A \cup B = \{(x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)) \,|\, x \in X\}$, *where*
$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$, *and*
$\nu_{A\cup B}(x) = \min\{\nu_A(x), \nu_B(x)\}$
$A + B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x)) \,|\, x \in X\}$, *where*
$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$, *and*
$\nu_{A+B}(x) = \nu_A(x) \cdot \nu_B(x)$
$A \cdot B = \{(x, \mu_{A\cdot B}(x), \nu_{A\cdot B}(x)) \,|\, x \in X\}$, *where*
$\mu_{A\cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$, *and*
$\nu_{A\cdot B}(x) = \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)$

In [2] it is proved that the operations $\cap$ and $\cup$ are commutative, associative, distributive among themselves, idempotent and satisfy De Morgan's law; the operations $+$ and $\cdot$ are commutative, associative, satisfy De Morgan's law, and are distributive with respect to $\cap$ and $\cup$.

To measure hesitancy of membership of an element to a intuitionistic fuzzy set, Atanassov [2] used a third function.

 ISBN: 978-1-61208-254-7

Definition 3. *Given an IFS $A = \{(x, \mu_A(x), \nu_A(x)) \,|\, x \in X\}$ over an non-empty universe of discourse X, the degree of indeterminacy of x to A is given by*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{4}$$

The function $\pi(x)$ is also called the `intuitionistic fuzzy index`, the `hesitancy`, or the `ignorance degree` of x to A. Clearly, $0 \leq \pi_A(x) \leq 1, \forall x \in X$. If $\pi_A(x) = 0, \forall x \in X$, then $\nu(x) = 1 - \mu(x)$ and the intuitionistic fuzzy set A is reduced to an ordinary fuzzy set A:

$$A = \{(x, \mu_A(x), 1 - \mu_A(x)) \,|\, x \in X\} \tag{5}$$

Some authors (Yusoff et al. [3], Zeng and Li [4]) consider that the third parameter $\pi(x)$ cannot be omitted from the definition of an AIFS:

$$A = \{(x, \mu_A(x), \nu_A(x), \pi_A(x)) \,|\, x \in X\} \tag{6}$$

and so an AIFS can be depicted as in Figure 2. A line parallel to the $\mu_A(x) + \nu_A(x) = 1$ diagonal describes a crisp set of elements x with the same level of hesitancy to A.

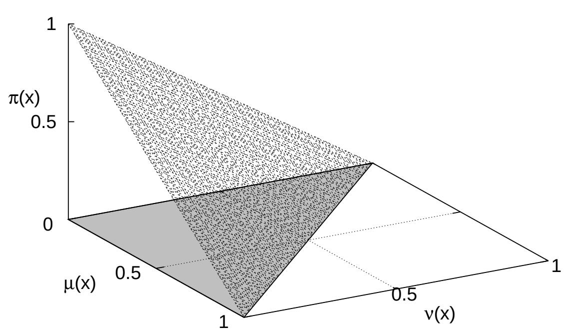

Fig. 2. AIFS with explicit fuzzy index: $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

IFSs are not a trivial generalization of ordinary Fuzzy Sets (FS) because they can be represented in the form $[A, B]$, where A and B are ordinary fuzzy sets or, even more, one can define modal operators *necessity* and *possibility* over IFS (see Atanassov [5]):

$$\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) \,| x \in X\} \tag{7}$$

$$\Diamond A = \{(x, 1 - \nu_A(x), \nu_A(x)) \,| x \in X\} \tag{8}$$

such that

$$\begin{aligned} \Box A \quad &\subset \quad A \quad \subset \quad \Diamond A && (9)\\ \Box \overline{A} \quad &= \quad \overline{\Diamond A} && (10)\\ \Diamond \overline{A} \quad &= \quad \overline{\Box A} && (11)\\ & && (12) \end{aligned}$$

while in ordinary fuzzy sets we have

$$\Box A = A = \Diamond A \tag{13}$$

Of course, all FS results can be easily generalized for IFS. Deschrijver and Kerre ([6], [7]) proved that AIFSs can also be seen as L-fuzzy sets in the sense of Goguen [8] by taking the lattice ${}_\star L = \{(x_1, x_2) \in [0, 1]^2 \,|\, x_1 + x_2 \leq 1\}$ with the partial order $\leq_{\star L}$ defined as

$$(x_1, x_2) \leq_{\star L} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \geq y_2$$

In [6] it is proved that $({}_\star L, \leq_{\star L})$ is a complete and bounded lattice with the smallest element $0_{\star L} = (0, 1)$ and the greatest element $1_{\star L} = (1, 0)$. This lattice (and the similar ones we'll introduce later in this section) can then be used to define intuitionistic fuzzy negation [9]:

Definition 4. *A function $\mathcal{N} : L \to L$, where $\mathcal{N}$ is strictly decreasing, continuous, and $\mathcal{N}(0_L) = 1_L$, $\mathcal{N}(1_L) = 0_L$ is called an intuitionistic fuzzy negation.*
$\mathcal{N}$ is a strong fuzzy negation if it is involutive, that is $\mathcal{N}(\mathcal{N}(x)) = x$ holds for all $x \in L$.

Recall that a function $\varphi : [0, 1] \to [0, 1]$ that is continuous and strictly increasing, such that $\varphi(0) = 0$ and $\varphi(1) = 1$, is called automorphism.

Deschrijver and Kerre [6] also proved that any strong intuitionistic fuzzy negation $\mathcal{N}$ is characterised by a strong negation $N : [0, 1] \to [0, 1]$ such that, for all $(x_1, x_2) \in L$, $\mathcal{N}(x_1, x_2) = (N(1-x_2), 1-N(x_1))$. Trillas et al. [10] proved that $N : [0, 1] \to [0, 1]$ is a strong negation if and only if there exists an automorphism φ of the unit interval such that $N(x) = \varphi^{-1}(1 - \varphi(x))$.

II. Generalised Intuitionistic Fuzzy Sets

In the sequel, let X be a non-empty set and let us consider $A = \{(x, \mu_A(x), \nu_A(x)) \,|\, x \in X\}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$, are used to define the degree of membership and the degree of non-membership, respectively, of x to A. Given an element $x \in X$, the condition $\mu(x) + \nu(x) \leq 1$ included in the definition of AIFSs suggests that if one of the two membership/non-membership functions has a big value (close to 1), the other function should have a very small value (close to 0) such that their sum is less than one. But it is possible that both functions have small values, that is membership degree and non-membership degree are quite insignificant. As one can see on Figure 1, both $\mu(x)$ and $\nu(x)$ have small values (less than 0.5) in the square with opposite corners $(0; 0)$ and $(0.5; 0.5)$ and only one of them has a big value (bigger than 0.5) in the two remaining triangles. The two cases are equal possible, in the sense that they cover surfaces of same size. The definition of an AIFS shows proneness to many generalisations. Atanassov's definition assumes that the membership and non-membership functions must have their sum smaller than or equal to one for every element of the universe of discourse. While it is a good hypothesis in many practical situations, there are cases when this constraint does not work and it must be replaced by other relations.

The definition of an AIFS shows proneness to many generalisations. A first extension was proposed by T. K. Mondal and S. K. Samanta [11], where the functions μ and ν satisfy the condition $\mu(x) \wedge \nu(x) \leq 0.5$, $\forall x \in X$. A second extension to both Atanassov and Mondal-Samanta models was proposed

 ISBN: 978-1-61208-254-7

by H.C. Liu [12], by using a constant $L \in [0,1]$ such that the functions μ and ν satisfy the condition $\mu(x)+\nu(x) \leq 1+L, \forall x \in X$ and $L \in [0,1]$.

The main contribution of this paper is the replacement of the original Atanassov relation by some other conditions. We think that both functions can take any values in $[0,1]$ as long as the ignorance degree of x to A is non-negative and less than or equal to one (after we reshape it in an appropriate way). The condition $0 \leq \mu_A(x)+\nu_A(x) \leq 1, \forall x \in X$ is just a choice and it can be replaced by others. If Atanassov's original definition dealt with the left bottom triangle of the unit square, we will consider all four right angle triangles in the unit square (with the right angle a corner of the square), plus some other combinations of them, obtained by combining triangles between the square's diagonals, as well as the inscribed circle in the square. Therefore, our first generalisation (GAIFS1) is given by:

Definition 5. *Let X be a non-empty universe of discourse. Then a generalised Atanassov intuitionistic fuzzy set (GAIFS1) is described by $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where the membership/non-membership functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the condition*

$$\mu(x)+\nu(x) \geq 1, \forall x \in X \tag{14}$$

The degree of indeterminacy of x to A is defined as

$$\pi_A(x) = \mu_A(x)+\nu_A(x)-1 \tag{15}$$

and, once again, clearly $0 \leq \pi_A(x) \leq 1, \forall x \in X$.
If $\pi_A(x) = 0, \forall x \in X$ then $\nu(x) = 1-\mu(x)$ and the intuitionistic fuzzy set A is reduced to an ordinary fuzzy set $A = \{(x, \mu_A(x), 1-\mu_A(x)) \mid x \in X\}$.

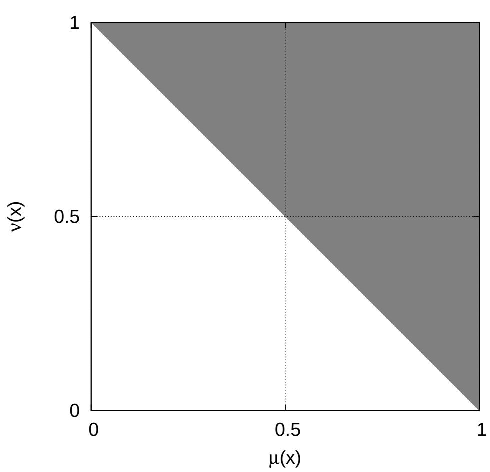

Fig. 3. First (GAIFS1) new definition of an AIFS

If we take the set

$$L^\star = \{(x_1,x_2) \in [0,1]^2 \mid x_1+x_2 \geq 1\}$$

with the partial order $\leq_{L^\star}$ defined as

$$(x_1,x_2) \leq_{L^\star} (y_1,y_2) \iff x_1 \geq y_1 \wedge x_2 \leq y_2$$

and , for each $A \subset L^\star$, we define:

$$\begin{aligned} \sup A &= (\inf\{x_1 \in [0,1] \mid \exists x_2 \in [0,1], (x_1,x_2) \in A\}, \\ &\quad \sup\{x_2 \in [0,1] \mid \exists x_1 \in [0,1], (x_1,x_2) \in A\}) \\ \inf A &= (\sup\{x_1 \in [0,1] \mid \exists x_2 \in [0,1], (x_1,x_2) \in A\}, \\ &\quad \inf\{x_2 \in [0,1] \mid \exists x_1 \in [0,1], (x_1,x_2) \in A\}) \end{aligned}$$

then $(L, \leq_{L^\star})$ is a complete lattice. The lattice can be defined as an algebraic structure $(L^\star, \wedge, \vee)$ where the meet and join operators are defined respectively

$$\begin{aligned} (x_1,x_2) \wedge (y_1,y_2) &= (\max(x_1,y_1), \min(x_2,y_2)) \\ (x_1,x_2) \vee (y_1,y_2) &= (\min(x_1,y_1), \max(x_2,y_2)) \end{aligned}$$

The smallest element is $0_{L^\star} = (1,0)$ and the greatest element is $1_{L^\star} = (0,1)$. Therefore, an GAIFS1 A is a L-fuzzy set whose L-membership function $\chi^A \in (L^\star)^X = \{\chi : X \to L^\star\}$ is defined such that for each $x \in X$, $\chi^A(x) = (\mu_A(x), \nu_A(x))$. The shaded area in Figure 4 is the set of elements $x = (x_1,x_2)$ belonging to $L^\star$.

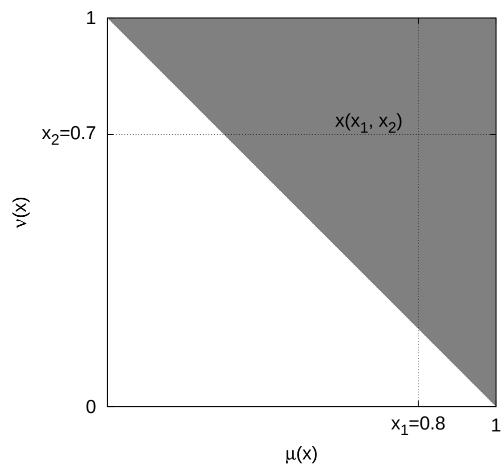

Fig. 4. New Intuitionistic Fuzzy Set as a L-fuzzy Set

The order $\leq_{L^\star}$ of $L^\star$ induces a natural partial order on $(L^\star)^X$: given $\chi^A, \chi^B \in (L^\star)^X$, we say that $\chi^A \leq_{L^\star} \chi^B$ if and only if $\chi^A(x) \leq_{L^\star} \chi^B(x)$ for all $x \in X$.
Thus, $((L^\star)^X, \leq_{L^\star})$ is a bounded and complete lattice in which the least and greatest elements are $\chi^{0_{L^\star}}$ and $\chi^{1_{L^\star}}$, respectively. Of course, $\chi^{0_{L^\star}}(x) = 0_{L^\star}$ and $\chi^{1_{L^\star}}(x) = 1_{L^\star}$, for all $x \in X$. The same considerations apply to all other L lattices we will define in the sequel. By using

$$A \subset B \text{ iff } \forall x \in X \; \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x)$$

all AIFS original operations can be re-written for GAIFS1.

The second generalisation (GAIFS2) is given by:

Definition 6. *Let X be a non-empty universe of discourse. Then a generalised Atanassov intuitionistic fuzzy set (GAIFS2) is described by $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where the membership/non-membership functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the condition*

$$\mu(x) \leq \nu(x), \forall x \in X \tag{16}$$

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The degree of indeterminacy of x to A is defined as

$$\pi_A(x) = \nu_A(x) - \mu_A(x) \tag{17}$$

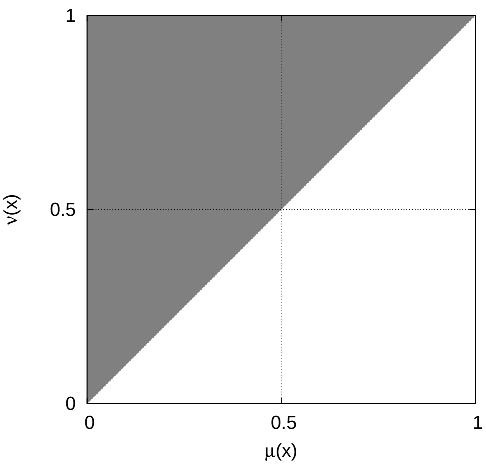

Fig. 5. Second (GAIFS2) new definition of an AIFS

The corresponding complete lattice in this case is

$$^{\star}L = \{(x_1, x_2) \in [0,1]^2 \,|\, x_1 \leq x_2\}$$

with the partial order $\leq_{^\star L}$ defined as

$$(x_1, x_2) \leq_{^\star L} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \leq y_2$$

As described by Deschrijver in [13], if $x = (x_1, x_2) \in {}^{\star}L$, then the length $x_2 - x_1$ is called the uncertainty and is denoted by x_π. The interval $[x_1, x_2]$ gives the "range between a pessimistic and an optimistic truth evaluation of a proposition" [13]. The smallest and the largest elements in $^{\star}L$ are $0_{^\star L} = (0, 0)$ and $1_{^\star L} = (1, 1)$, respectively. By using

$$A \subset B \text{ iff } \forall x \in X \; \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x)$$

all AIFS original operations can be re-written for GAIFS2. The third generalisation (GAIFS3) is given by:

Definition 7. *Let X be a non-empty universe of discourse. Then a generalised Atanassov intuitionistic fuzzy set (GAIFS3) is described by $A = \{(x, \mu_A(x), \nu_A(x)) \,|\, x \in X\}$, where the membership/non-membership functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ satisfy the condition*

$$\mu(x) \geq \nu(x), \; \forall x \in X \tag{18}$$

The degree of indeterminacy of x to A is defined as

$$\pi_A(x) = \mu_A(x) - \nu_A(x) \tag{19}$$

The corresponding complete lattice in this case is

$$L_\star = \{(x_1, x_2) \in [0,1]^2 \,|\, x_1 \geq x_2\}$$

with the partial order $\leq_{L_\star}$ defined as

$$(x_1, x_2) \leq_{L_\star} (y_1, y_2) \iff x_1 \geq y_1 \wedge x_2 \geq y_2$$

The interval $[x_2, x_1]$ gives, once again, the "range between a pessimistic and an optimistic truth evaluation of a proposition", as stated in [13]. The smallest and the largest elements in $L_\star$ are $0_{L_\star} = (1, 1)$ and $1_{L_\star} = (0, 0)$, respectively.

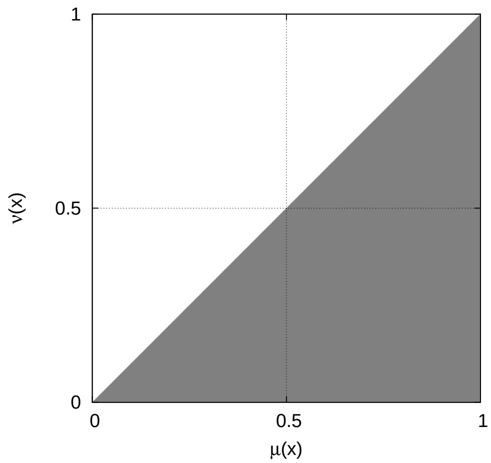

Fig. 6. Third (GAIFS3) new definition of an AIFS

By using

$$A \subset B \text{ iff } \forall x \in X \; \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x)$$

all AIFS original operations can be re-written for GAIFS3.

The fourth generalisation (GAIFS4) is given by:

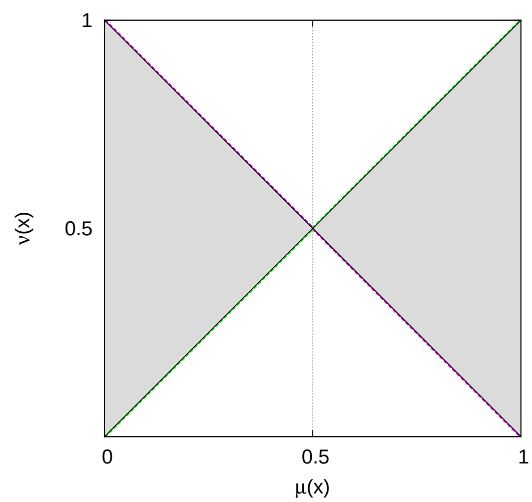

Fig. 7. Fourth (GAIFS4) new definition of an AIFS

Definition 8. *Let X be a non-empty universe of discourse. Then a generalised Atanassov intuitionistic fuzzy set (GAIFS4) is described by $A = \{(x, \mu_A(x), \nu_A(x)) \,|\, x \in X\}$, where the membership/non-membership functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ satisfy the condition*

$$\begin{gathered} \mu(x) \geq \nu(x), \;\; \textit{and} \;\; \mu(x) + \nu(x) \geq 1, \;\; \textit{or} \\ \mu(x) \leq \nu(x), \;\; \textit{and} \;\; \mu(x) + \nu(x) \leq 1, \forall x \in X \end{gathered} \tag{20}$$

 ISBN: 978-1-61208-254-7

The fifth generalisation (GAIFS5) is given by:

Definition 9. *Let X be a non-empty universe of discourse. Then a generalised Atanassov intuitionistic fuzzy set (GAIFS5) is described by $A = \{(x, \mu_A(x), \nu_A(x)) \,|\, x \in X\}$, where the membership/non-membership functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the condition*

$$\mu(x) \le \nu(x), \;\; \textit{and} \;\; \mu(x) + \nu(x) \ge 1, \;\; \textit{or}$$
$$\mu(x) \ge \nu(x), \;\; \textit{and} \;\; \mu(x) + \nu(x) \le 1, \forall x \in X \tag{21}$$

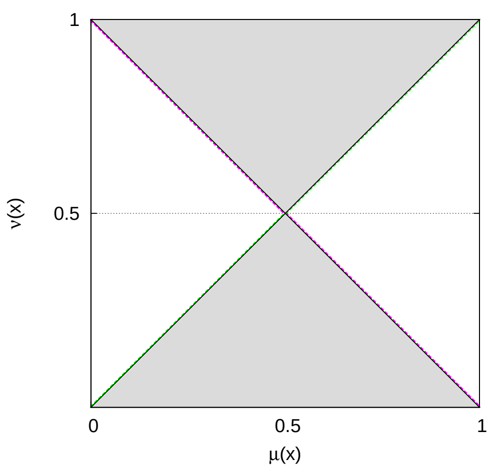

Fig. 8. Fifth (GAIFS5) new definition of an AIFS

It is also possible to consider the case when functions μ and ν cannot take values in the neighbourhoods of the four corners of the square $[0,1]^2$, that is giving our sixth generalization (GAIFS6):

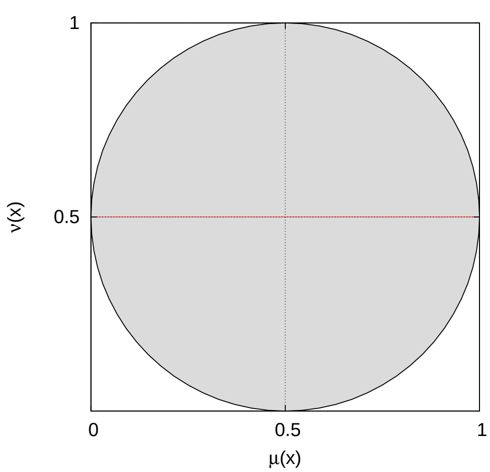

Fig. 9. Sixth (GAIFS6) new definition of an AIFS

Definition 10. *Let X be a non-empty universe of discourse. Then a generalised Atanassov intuitionistic fuzzy set (GAIFS6) is described by $A = \{(x, \mu_A(x), \nu_A(x)) \,|\, x \in X\}$, where the membership/non-membership functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the condition*

$$\mu^2(x) + \nu^2(x) \le 1, \forall x \in X \tag{22}$$

III. Generalised Interval-valued Fuzzy Sets

Interval-valued fuzzy sets were introduced by Zadeh [14], Grattan-Guiness [15], Jahn [16], and Sambuc [17]. Because it is hard in real life problems to assign a precise membership degree to elements in fuzzy sets, this was replaced by an interval $[\mu_1, \mu_2]$, with $0 \le \mu_1 \le \mu_2 \le 1$ to which the membership degree belongs. The length of the interval is a measure of uncertainty of the membership of an element $x \in X$ to an interval-valued fuzzy set (IVFS) A. It is similar to the degree of indeterminacy of x to A in AIFS. The FS (AIFS) standard operations (union, intersection, complementation) can be extended to IVFS in the canonical way. If $M = [\mu_1, \mu_2]$ and $N = [\nu_1, \nu_2]$ are two IVFS, then for all $x \in X$

$$(M \cap N)(x) = [\min(\mu_1(x), \nu_1(x)), \min(\mu_2(x), \nu_2(x))] \tag{23}$$
$$(M \cup N)(x) = [\max(\mu_1(x), \nu_1(x)), \max(\mu_2(x), \nu_2(x))] \tag{24}$$
$$\bar{M}(x) = [1 - \mu_2(x), 1 - \mu_1(x)] \tag{25}$$

The equivalence between AIFS and IVFS has been studied in [18] and [6]. If $A = [\mu_1, \mu_2]$, $0 \le \mu_1 \le \mu_2 \le 1$ then $\mu_1 - \mu_2 \le 0$ so $\mu_1 + 1 - \mu_2 \le 1$. By defining $\mu_1 = \mu$ and $\nu = 1 - \mu_2$, we obtain an AIFS. Conversely, starting with an AIFS $A = (\mu, \nu), \mu + \nu \le 1$, we can create the interval $[\mu, 1 - \nu]$ to correspond to an IVFS.

In the case of our first generalisation (GAIFS1), where $\mu(x) + \nu(x) \ge 1$ for all $x \in X$, the above equivalence still holds. If $A = [\mu_1, \mu_2]$, $0 \le \mu_1 \le \mu_2 \le 1$ then $0 \le \mu_2 - \mu_1$ so $1 \le \mu_2 + 1 - \mu_1$ and, by defining $\mu = 1 - \mu_1$ and $\nu = \mu_2$, we obtain an AIFS.

The above equivalence also holds trivially for GAIFS2, GAIFS3, GAIFS4, and GAIFS5 generalizations. For instance, in the case of GAIFS2, $\mu(x) \le \nu(x)$, so the corresponding IVFS should be characterised by $[\mu, \nu]$. In the case of GAIFS3, $\nu(x) \le \mu(x)$ holds, so the corresponding IVFS should be characterised by $[\nu, \mu]$, etc. For GAIFS6 case, we take the IVFS to be given by the interval $[\mu, 1-\nu]$. Then $\mu^2 \le (1-\nu)^2$ and $\mu^2+\nu^2 \le (1-\nu)^2+\nu^2 \le 1-2\nu+2\nu^2 \le 1+2\nu(\nu-1) \le 1$.

IV. Automorphisms

We deal with our first generalization GAIFS1 only, the other cases are treated in a similar way. GAIFS1 is equivalent to the lattice $(L^\star, \wedge, \vee, 0_{L^\star}, 1_{L^\star})$, where

$$L^\star = \{(x_1, x_2) \in [0,1]^2 \,|\, x_1 + x_2 \ge 1\}$$

with the partial order $\le_{L^\star}$ defined as

$$(x_1, x_2) \le_{L^\star} (y_1, y_2) \iff x_1 \ge y_1 \wedge x_2 \le y_2$$

and $0_{L^\star} = (1, 0)$ and $1_{L^\star} = (0, 1)$. The operations are defined as

$$(x_1, x_2) \wedge (y_1, y_2) = (x_1 \vee y_1, x_2 \wedge y_2) \tag{26}$$
$$(x_1, x_2) \vee (y_1, y_2) = (x_1 \wedge y_1, x_2 \vee y_2) \tag{27}$$

and one can easily verify that

$$(x_1, x_2) \wedge (1, 0) = (x_1 \vee 1, x_2 \wedge 0) = (1, 0) = 0_{L^\star}$$
$$(x_1, x_2) \vee (1, 0) = (x_1 \wedge 1, x_2 \vee 0) = (x_1, x_2)$$
$$(x_1, x_2) \wedge (0, 1) = (x_1 \vee 0, x_2 \wedge 1) = (x_1, x_2)$$
$$(x_1, x_2) \vee (0, 1) = (x_1 \wedge 0, x_2 \vee 1) = (0, 1) = 1_{L^\star}$$

As in any lattice, the meet operator $\wedge$ and the join operator $\vee$ are related to the ordering $\leq_{L^\star}$ by the following equivalences: for every $x = (x_1, x_2), y = (y_1, y_2) \in L^\star$

$$x \leq_{L^\star} y \iff x \vee y = y \iff x \wedge y = x$$

Indeed, if $(x_1, x_2) \leq_{L^\star} (y_1, y_2)$, then

$$(x_1, x_2) \vee (y_1, y_2) = (x_1 \wedge y_1, x_2 \vee y_2) = (y_1, y_2)$$
$$(x_1, x_2) \wedge (y_1, y_2) = (x_1 \vee y_1, x_2 \wedge y_2) = (x_1, x_2)$$

Definition 11. *An* `automorphism` *of $L^\star$ is a bijection $f : L^\star \to L^\star$ such that $f(x) \leq_{L^\star} f(y)$ if and only if $x \leq_{L^\star} y$, for all $x, y \in L^\star$.*
An `anti-automorphism` *of $L^\star$ is a bijection $f : L^\star \to L^\star$ such that $f(x) \leq_{L^\star} f(y)$ if and only if $y \leq_{L^\star} x$, for all $x, y \in L^\star$.*

It is obvious that an automorphism takes $0_{L^\star}$ and $1_{L^\star}$ to themselves, while an anti-automorphism interchanges these elements.

We denote by $\mathbf{A}ut(L^\star)$ the set of all automorphisms of $L^\star$ and by $\mathbf{M}ap(L^\star)$ the set of all automorphisms and anti-automorphisms of $L^\star$. They are groups under the composition of morphisms and $\mathbf{A}ut(L^\star)$ is a normal subgroup of order 2 of $\mathbf{M}ap(L^\star)$. [19]

Let $f \in \mathbf{A}ut(L^\star)$. Since $f(x) \leq_{L^\star} f(y)$ if and only if $x \leq_{L^\star} y$, for all $x, y \in L^\star$, then

$$f(x \vee y) = f(x) \vee f(y)$$
$$f(x \wedge y) = f(x) \wedge f(y)$$
$$f((1, 0)) = (1, 0)$$
$$f((0, 1)) = (0, 1)$$

If f is an automorphism of $[0, 1]$, then, for $(x_1, x_2) \in L^\star$, $(x_1, x_2) \mapsto (f(x_1), f(x_2))$ is an automorphism of $L^\star$.

An anti-automorphism $\mathcal{N}$ such that $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^\star$ is an `involution` or `negation`. Obviously, $\mathcal{N}(0_{L^\star}) = 1_{L^\star}$ and $\mathcal{N}(1_{L^\star}) = 0_{L^\star}$.

All elements but identity of $\mathbf{A}ut(L^\star)$ are of infinite order; all anti-automorphisms are of infinite order or of order two. The order two anti-automorphisms are `involutions` and their set is denoted by $\mathbf{I}nv(L^\star)$. One (classical) involution is $\alpha : L^\star \to L^\star$ given by $\alpha(x_1, x_2) = (1 - x_2, 1 - x_1)$, all other involutions are of the form $f^{-1}\alpha f$, for any $f \in \mathbf{M}ap(L^\star)$.

In [20] it is proved that if $\mathcal{N}$ is an involutive negator on $L^\star$ (negation) and $N : [0, 1] \to [0, 1]$, $N(a) = pr_1\mathcal{N}(a, 1 - a)$, for all $a \in [0, 1]$, then $\mathcal{N}(x_1, x_2) = (N(1 - x_2), 1 - N(x_1))$, for all $(x_1, x_2) \in L^\star$.

A triangular norm on $L^\star$ (`t-norm`) is any increasing, commutative, associative mapping $T : L^\star \times L^\star \to L^\star$ satisfying $T(1_{L^\star}, x) = x$ for all $x \in L^\star$. A triangular co-norm on $L^\star$ (`t-conorm`) is any increasing, commutative, associative mapping $S : L^\star \times L^\star \to L^\star$ satisfying $S(0_{L^\star}, x) = x$ for all $x \in L^\star$.

V. Conclusion

We introduced six possible new definitions for intuitionistic fuzzy sets by challenging the base condition in Atanassov's definition. While keeping the two membership functions, we extended the range of possible combinations between them and showed some interesting properties. We intend to further develop the approach for measuring similarity and compatibility between different sorts of intuitionistic fuzzy sets.

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 ISBN: 978-1-61208-254-7