Reduction of Decoherence in Quantum Information Systems Using Direct Adaptive Control of Infinite Dimensional Systems

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***Abstract*— Quantum systems are inherently infinite dimensional. In particular quantum computers will use quantum systems as gates to store and manipulate information. But such systems suffer from decoherence which is caused by the quantum gate becoming entangled with its environment and losing information into that quantum environment. Feedback control has the promise of reducing this decoherence, but the feedback must be adaptive in the sense that it can perform its control tasks with very little information about the details of the quantum system itself. This paper is concerned with providing a framework for adaptive control of infinite dimensional quantum systems. The quantum system is described as a linear continuous-time infinite-dimensional plant on a complex Hilbert space with persistent disturbances of known waveform but unknown amplitude and phase caused by fluctuations in the external quantum environment. We show here that there is a stabilizing direct model reference adaptive control law with disturbance rejection and robustness properties. The plant is described by a closed, densely defined linear operator, which is the Hamiltonian of the quantum system that generates a continuous semigroup of bounded operators on the complex Hilbert space of states. There is no state or disturbance estimation used in this adaptive approach. We show that adaptive control can produce convergence of a quantum system to a Decoherence-Free Subspace. Our research direction continues on using our developing research in adaptive control of infinite dimensional systems to explore how these feedback control ideas in conjunction with quantum gates and quantum error correction can reduce decoherence in quantum information and computing.**

Keywords - Quantum Systems; Adaptive Control; Infinite-Dimensional Systems.

I. INTRODUCTION

Quantum systems are inherently infinite dimensional. In particular quantum computers will use quantum systems as gates to store and manipulate information. But such systems suffer from decoherence which is caused by the quantum gate becoming entangled with its environment and losing information into that quantum environment. Feedback control has the promise of reducing this decoherence, but the feedback must be adaptive in the sense that it can perform its control tasks with very little information about the details of the quantum system itself. This paper is concerned with providing a framework for adaptive control of infinite dimensional quantum systems.

The quantum system is described as a linear continuous-time infinite-dimensional plant on a complex Hilbert space with persistent disturbances of known waveform but unknown amplitude and phase caused by fluctuations in the external quantum environment. We show here that there is a stabilizing direct model reference adaptive control law with disturbance rejection and robustness properties. The plant is described by a closed, densely defined linear operator, which is the Hamiltonian of the quantum system that generates a continuous semigroup of bounded operators on the complex Hilbert space of states. There is no state or disturbance estimation used in this adaptive approach.

Our overall direction is on using our developing research in adaptive control of infinite dimensional systems to explore how these feedback control ideas in conjunction with quantum gates and quantum error correction can reduce decoherence in quantum information and computing.

Let X be an infinite dimensional separable complex Hilbert space with inner product (x, y) and corresponding norm $\|x\| \equiv \sqrt{(x,x)}$.

Consider the Linear Infinite Dimensional Plant with *Persistent Disturbances*:

$$\begin{cases} \frac{\partial}{\partial t}x(t) = Ax(t) + Bu(t) + \Gamma\, u_D(t) \\ x(0) \equiv x_0 \in D(A) \subseteq X \\ Bu \equiv \sum_{i=1}^{m} b_i u_i \\ y(t) = Cx(t) + Eu_D(t) \\ y_i \equiv (c_i, x(t)), i = 1 \ldots m \end{cases} \tag{1}$$

where $x \in D(A)$ is the plant state, $b_i \in D(A)$ are actuator influence functions, $c_i \in D(A)$ are sensor influence functions, $u, y \in \Re^m$ are the control input and plant output m-vectors respectively, u_D is a disturbance with known basis functions φ_D . The persistent disturbances u_D will enter the plant through the state channels Γ and the output channels E .

In order to accomplish disturbance rejection in a direct

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adaptive scheme, we will make use of a definition, given in [4] and [7], for persistent disturbances:

Definition: A disturbance vector $u_D \in R^q$ is said to be persistent if it satisfies the disturbance generator equations:

$$\begin{cases} u_D(t) = \theta z_D(t) \\ \dot{z}_D(t) = F z_D(t) \end{cases} \text{or} \begin{cases} u_D(t) = \theta z_D(t) \\ z_D(t) = L\varphi_D(t) \end{cases} \tag{2}$$

where F is a marginally stable matrix and $\varphi_D(t)$ is a vector of known functions forming a basis for all the possible disturbances. This is known as "a disturbance with known waveform but unknown amplitudes". We can easily show that an operator L exists to relate the persistent disturbances to a known basis vector $\varphi_D(t)$, but the adaptive controller will not need to know the actual L.

The *objective* of control in this paper will be to cause the output $y(t)$ of the plant to regulate asymptotically:

$$y \xrightarrow[t\to\infty]{} 0 \tag{3}$$

and this control objective will be accomplished by a *Direct Adaptive Control Law* of the form:

$$u = G_e y + G_D \varphi_D \tag{4a}$$

The direct adaptive controller will have adaptive gains given by:

$$\begin{cases} \dot{G}_e = -yy^* \gamma_e ; \gamma_e > 0 \\ \dot{G}_D = -y\varphi_D^* \gamma_D ; \gamma_D > 0 \end{cases} \tag{4b}$$

Note that the output feedback gains are directly adapted and no estimation or identification of plant information is used in the control law.

II. Ideal Trajectories

We define the Ideal Trajectories for (1) the following way:

$$\begin{cases} x_* = S_1 z_D \\ u_* = S_2 z_D \end{cases} \text{with } z_D \in \Re^{N_D} \tag{5}$$

where the ideal trajectory $x_*(t)$ is generated by the ideal control $u_*(t)$ from

$$\begin{cases} \dfrac{\partial x_*}{\partial t} = Ax_* + Bu_* + \Gamma u_D \\ y_* = Cx_* + Eu_D = 0 \end{cases} \tag{6}$$

If such ideal trajectories exist, they will be linear combinations of disturbance state, and they will produce exact output tracking in a disturbance-free plant (8).

By substitution of (5) into (6), we obtain the *Model Matching Conditions*:

$$\begin{cases} AS_1 + BS_2 = S_1 F + \underbrace{H_1}_{\Gamma\theta} \\ CS_1 = H_2 = -E\theta \end{cases} \tag{7}$$

where $S_1 : \Re^{N_D} \to D(A) \subset X, S_2 : \Re^{N_D} \to \Re^M$.

Because (S_1, S_2) are both of finite rank, they are bounded linear operators on their respective domains.

A Special Case occurs when E=0 and $Range(\Gamma) \subseteq Range(B)$. Then there exists S2 such that $BS_2 + \Gamma\theta = 0$ and S1=0. In this case the full system state x becomes disturbance-free, but in general we really only want to make the output y disturbance-free.

III. Normal Form

We need two lemmas:

Lemma 1: If CB is nonsingular then $P_1 \equiv B(CB)^{-1}C$ is a (non-orthogonal) bounded projection onto the range of B, R(B), along the null space of C, N(C) with $P_2 \equiv I - P_1$ the complementary bounded projection, and $X = R(B) \oplus N(C)$, as well as $D(A) = R(B) \oplus [N(C) \cap D(A)]$.

Proof of Lemma 1: See [17].

Now for the above pair of projections $(P_1.P_2)$ we have

$$\begin{cases} \dfrac{\partial P_1 x}{\partial t} = P_1 \dfrac{\partial x}{\partial t} = \underbrace{(P_1 A P_1)}_{A_{11}} P_1 x + \underbrace{(P_1 A P_2)}_{A_{12}} P_2 x + \underbrace{(P_1 B)}_{B} u \\ \dfrac{\partial P_2 x}{\partial t} = P_2 \dfrac{\partial x}{\partial t} = \underbrace{(P_2 A P_1)}_{A_{21}} P_1 x + \underbrace{(P_2 A P_2)}_{A_{22}} P_2 x + \underbrace{(P_2 B)}_{=0} u \\ y = \underbrace{(CP_1)}_{C} P_1 x + \underbrace{(CP_2)}_{=0} P_2 x \end{cases}$$

which implies

$$\begin{cases} \dfrac{\partial P_1 x}{\partial t} = A_{11} P_1 x + A_{12} P_2 x + Bu \\ \dfrac{\partial P_2 x}{\partial t} = A_{21} P_1 x + A_{22} P_2 x \\ y = CP_1 x = Cx \end{cases}$$

Because

$$y = Cx = C(B(CB)^{-1}C)x = CP_1 x$$

and $P_1 x = B(CB)^{-1}Cx = B(CB)^{-1} y$

and $CP_2 = C - CB(CB)^{-1}C = 0$

and $P_2 B = B - B(CB)^{-1}CB = 0.$

Lemma 2: If CB is nonsingular, then there exists and invertible, bounded linear operator $W \equiv \begin{bmatrix} C \\ W_2 P_2 \end{bmatrix} : X \to \tilde{X} \equiv R(B) x l_2$ such that

$\overline{B} \equiv WB = \begin{bmatrix} CB \\ 0 \end{bmatrix}, \overline{C} \equiv CW^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix}$, and $\overline{A} \equiv WAW^{-1}$.

This coordinate transformation puts (1) into normal form

$$\begin{cases} \dot{y} = \overline{A}_{11} y + \overline{A}_{12} z_2 + CBu \\ \dfrac{\partial z_2}{\partial t} = \overline{A}_{21} y + \overline{A}_{22} z_2 \end{cases} \quad (8)$$

where the subsystem: $(\overline{A}_{22}, \overline{A}_{12}, \overline{A}_{21})$ is called the zero dynamics of (1) and

$$\overline{A}_{11} \equiv CA_{11}B(CB)^{-1} = CAB(CB)^{-1}; \overline{A}_{12} \equiv CAW_2^*;$$

$$\overline{A}_{21} \equiv W_2 A_{21} B(CB)^{-1}; \overline{A}_{22} \equiv W_2 A_{22} W_2^*$$

and $W_2 : X \to l_2$ by $W_2 x \equiv \begin{bmatrix} (\theta_1, P_2 x) \\ (\theta_2, P_2 x) \\ (\theta_3, P_2 x) \\ \dots \end{bmatrix}$ is an isometry from $N(C)$ into l_2.

Proof of Lemma 2: See [17].

Now we have the following theorem about the *Existence of Ideal Trajectories*:

Theorem 1: Assume CB is nonsingular. Then

$$\sigma(F) = \sigma_p(F) \subset \rho(\overline{A}_{22})$$
$$\equiv \{\lambda \in C / (\lambda I - \overline{A}_{22})^{-1} : l_2 \to l_2 \text{ is a bounded linear operator}\}$$

(or $\sigma_p(F) \cap \sigma(\overline{A}_{22}) = \varphi$ where $\sigma(\overline{A}_{22}) \equiv [\rho(\overline{A}_{22})]^c$), if and only there exist unique bounded linear operator solutions (S_1, S_2) satisfying the Matching Conditions (7).

Proof: See [17].

It is possible to relate the point spectrum $\sigma_p(\overline{A}_{22}) \equiv \left\{\lambda / \lambda I - \overline{A}_{22} \text{ not 1-1}\right\}$ to the set Z of transmission (or blocking) zeros of (A, B, C).

Similar to the finite-dimensional case [16], we can see that

$$Z \equiv \left\{ \begin{matrix} \lambda / V(\lambda) \equiv \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix}: \\ D(A)x\Re^m \to Xx\Re^m \text{ linear operator is not 1-1} \end{matrix} \right\}$$

Lemma 3: $Z = \sigma_p(\overline{A}_{22}) \equiv \left\{\lambda / \lambda I - \overline{A}_{22} \text{ is not 1-1}\right\}$ is called the point spectrum of $\overline{A}_{22}$. So, the transmission zeros of the infinite-dimensional open-loop plant (A, B, C) are the point spectrum of its zero dynamics $(\overline{A}_{22}, \overline{A}_{12}, \overline{A}_{21})$.

Proof of Lemma 3:

From

$$\overline{V}(\lambda) = \begin{bmatrix} \lambda I - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix} \underbrace{\begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix}}_{V(\lambda)} \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}$$

we obtain $\begin{bmatrix} \lambda I - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}$ not 1-1 if and only if $\begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix}$ not 1-1.

But, using normal form from Lemma 2,

$$\overline{V}(\lambda) \equiv \begin{bmatrix} \lambda I - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} \lambda I - \overline{A}_{11} & -\overline{A}_{12} & CB \\ -\overline{A}_{21} & \lambda I - \overline{A}_{22} & 0 \\ I_m & 0 & 0 \end{bmatrix}$$

And, therefore, $0 = \overline{V}(\lambda) h = \overline{V}(\lambda) \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$, if and only if

$$h_1 = 0; h_3 = (CB)^{-1} \overline{A}_{12} h_2; (\lambda I - \overline{A}_{22}) h_2 = 0.$$

So, $h \neq 0$, if and only if $h_2 \neq 0$. Therefore $\begin{bmatrix} sI - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}$ is not 1-1 if and only if $\lambda \in \sigma_p(\overline{A}_{22})$.

This completes the proof of Lemma 3.

Using Lemma 3 and Theorem 1, we have the following *Internal Model Principle*:

Corollary 1: Assume CB is nonsingular and $\sigma(\overline{A}_{22}) = \sigma_p(\overline{A}_{22}) = \sigma_p(P_2 A P_2)$ where $\overline{A}_{22} \equiv W_2^* P_2 A P_2 W_2$. There exist unique bounded linear operator solutions (S_1, S_2) satisfying the Matching Conditions (7) if and only if $\sigma(F) \cap Z = \varphi$, i.e., no eigenvalues of F can be zeros of the open-loop plant (A, B, C).

Note: $\lambda I - \overline{A}_{22}$ is not 1-1 if and only if there exists $x \neq 0$ such that $P_2 x \neq 0$ and

$$0 = (\lambda I - \overline{A}_{22}) W_2 P_2 x$$
$$= (\lambda \underbrace{W_2 W_2^*}_{I} - W_2 P A P_2 W_2^*) W_2 P_2 x$$
$$= [W_2 (\lambda I - P_2 A P_2) W_2^*] W_2 P_2 x$$

if and only if $W_2 (\lambda I - P_2 A P_2) W_2^*$ is not 1-1 on $N(C)$.

But W_2 is an isometry on $N(C)$.

Therefore $\sigma_p(\overline{A}_{22}) = \sigma_p(P_2 A P_2)$.

IV. STABILITY OF THE ERROR SYSTEM

The error system can be found from (1), (2) and (6): Define $e \equiv x - x_*$ and $\Delta u \equiv u - u_*$, this implies

$$\begin{cases} \frac{\partial e}{\partial t} = Ae + B\Delta u \\ y = y - 0 = \Delta y \equiv y - y_* = Ce \end{cases} \tag{9}$$

Now we consider the definition of Strict Dissipativity for infinite-dimensional systems and the general form of the "adaptive error system" to prove stability. The main theorem of this section will later be utilized to assess the convergence and stability of the adaptive controller with disturbance rejection for linear diffusion systems.

Noting that there can be some ambiguity in the literature with the definition of strictly dissipative systems, we modify the suggestion of Wen in [8] for finite dimensional systems and expand it to include infinite dimensional systems.

Definition 1: *The triple (A_c, B, C) is said to be **Strictly Dissipative (SD)** if A_c is a densely defined ,closed operator on $D(A_c) \subseteq X$ a complex Hilbert space with inner product (x, y) and corresponding norm $\|x\| \equiv \sqrt{(x,x)}$ and generates a C_0 semigroup of bounded operators $U(t)$,and (B,C) are bounded finite rank input/output operators with rank M where $B: R^m \to X$* and *$C: X \to R^m$. In addition there exist symmetric positive bounded operator P and Q on X such that*

$$0 \le p_{\min}\|e\|^2 \le (Pe,e) \le p_{\max}\|e\|^2; 0 \le q_{\min}\|e\|^2 \le (Qe,e) \le q_{\max}\|e\|^2$$

i.e. P,Q are bounded and coercive, and

$$\begin{cases} \mathrm{Re}(PA_c e, e) \equiv \frac{1}{2}[(PA_c e, e) + \overline{(PA_c e, e)}] \\ = \frac{1}{2}[(PA_c e, e) + (e, PA_c e)] \\ = -(Qe,e) \le -q_{\min}\|e\|^2; e \in D(A_c) \\ PB = C^* \end{cases} \tag{10}$$

where W^* is the adjoint of the operator W.

We also say that (A,B,C) is *Almost Strictly Dissipative (ASD)* when there exists $G_* mxm$ gain such that (A_c, B, C) is SD with $A_c \equiv A + BG_*C$. Note that if $P = I$ in (10)by the Lumer-Phillips Theorem [10], p405, we would have $\|U_c(t)\| \le e^{-\sigma t}; t \ge 0; \sigma \equiv q_{\min} > 0$.

Henceforth, we will make the following set of assumptions:

Hypothesis 1: *Assume the following:*

1) There exists a gain G_e^* such that the triple $(A_C \equiv A + BG_e^*C, B, C)$ is SD, i.e. (A,B,C) is ASD.

2) A is a densely defined ,closed operator on $D(A) \subseteq X$ and generates a C_0 semigroup of bounded operators $U(t)$,

3) φ_D is bounded.

From (5), we have $u_* = S_2 z_D$ and using (4a), we obtain:

$$\begin{aligned} \Delta u &\equiv u - u_* = (G_e y + G_D\varphi_D) - (S_2 \underbrace{z_D}_{L\varphi_D}) \\ &= G_e^* y + \Delta G_e y + \Delta G_D \varphi_D = G_e^* e_y + \Delta G \eta \end{aligned} \tag{11}$$

where

$\Delta G \equiv G - G_*; G \equiv \begin{bmatrix} G_e & G_D \end{bmatrix}; G_* \equiv \begin{bmatrix} G_e^* & S_2 L \end{bmatrix}; G_D^* \equiv S_2 L;$

and $\eta \equiv \begin{bmatrix} y \\ \varphi_D \end{bmatrix}$.

From (4), (9), and (11), the *Error System* becomes

$$\begin{cases} \frac{\partial e}{\partial t} = \underbrace{(A + BG_e^*C)}_{A_c} e + B\Delta G\eta = A_c e + B\rho; \\ e \in D(A); \rho \equiv \Delta G\eta \\ e_y = Ce \\ \Delta\dot{G} = \dot{G} - \dot{G}_* = \dot{G} = -e_y \eta^* \gamma \end{cases} \tag{12}$$

where $\gamma \equiv \begin{bmatrix} \gamma_e & 0 \\ 0 & \gamma_D \end{bmatrix} > 0$.

Since B,C are finite rank operators, so is BG_e^*C. Therefore $A_c \equiv A + BG_e^*C$ which has $D(A_c) = D(A)$, and generates a C_0 semigroup $U_c(t)$ because A does, see [9]Theo 2.1 p 497. Furthermore, by Theo 8.10 p 157 in [11], $x(t)$ remains in $D(A)$ and is differentiable there for all $t \ge 0$. This is because $F(t) \equiv B\rho = B\Delta G\eta$ is continuously differentiable in $D(A)$.

We see that (12) is the *feedback interconnection* of an infinite-dimensional linear subsystem with $e \in D(A) \subseteq X$ and a finite-dimensional subsystem with $\Delta G \in \Re^{mxm}$. This can be written in the following form using $w \equiv \begin{bmatrix} e \\ \Delta G \end{bmatrix} \in D \equiv D(A)x\Re^{mxm} \subseteq \bar{X} \equiv Xx\Re^{mxm}$:

$$\begin{cases} \frac{\partial w}{\partial t} = w_t = f(t,w) \equiv \begin{bmatrix} A_c e + B\rho(t) \\ -e_y \eta^* \gamma \end{bmatrix} \\ w(t_0) = w_0 \in D \text{ dense in } \bar{X} \equiv Xx\Re^{mxm} \end{cases} \tag{13}$$

The inner product on $\bar{X} \equiv Xx\Re^{mxm}$ can be defined as $(w_1, w_2) \equiv \left(\begin{bmatrix} x_1 \\ \Delta G_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ \Delta G_2 \end{bmatrix} \right) \equiv (x_1, x_2) + tr(\Delta G_2 \Delta G_1^*)$, which will make it a Hilbert space also.

Now, we present a new version of Barbalat-Lyapunov for systems on an infinite dimensional Hilbert space:

Theorem 2 (Lyapunov-Barbalat):

Let $w(t) = w(t, t_0, w_0) \in D$ and $V(t, w)$ satisfy:

$$\begin{cases} \alpha \|w\|^2 \le V(t,w) \le \beta \|w\|^2 \\ \dot{V}(t,w) \equiv \dfrac{\partial V(t,w)}{\partial t} + \dfrac{\partial V(t,w)}{\partial w} f(t,w) \le -S(w) \le 0 \end{cases}$$

for all $w \in D$. Then, *w(t)* is bounded in $\bar{X}$. Furthermore, if the following are true:

1) $S(w) \ge \mu \|\aleph w\|^2 \ \forall w \in D; \mu > 0$; with $\aleph$ a bounded operator on $D \subseteq \bar{X} \equiv X x \Re^{mxm} \to X$ such that $(\aleph w)_t = \aleph w_t$.

2) $\mathrm{Re}\left(\aleph w, \aleph f(t,w)\right)$ is bounded on bounded sets of $w \in D$, then $\aleph w(t) \xrightarrow[t\to\infty]{} 0.$

Proof: See Appendix I in [17].

For this proof, we will need the following version of Barbalat's Lemma; see [15] pp210-211:

Lemma 4: We say $f(t)$ is a *uniformly continuous* function on $(0,\infty)$ when for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that $|f(t_2) - f(t_1)| < \varepsilon \ \forall |t_2 - t_1| < \delta$. If $f(t)$ is a real, *uniformly continuous* function on $(0,\infty)$ with $\int_0^\infty f(t)dt < \infty$, then $f(t) \xrightarrow[t\to\infty]{} 0$.

Now we can prove the stability and convergence of the direct adaptively controlled error system (12):

Theorem 3: Under Hypothesis 1and $\mathrm{Re}(A_c e, e)$ bounded on bounded sets of $e \in D(A)$, we will have state and output tracking of the reference model: $e \xrightarrow[t\to\infty]{} 0$, and since C is a bounded linear operator: $e_y = y - y_m = Ce \xrightarrow[t\to\infty]{} 0$ with bounded adaptive gains $G \equiv \begin{bmatrix} G_e & G_m & G_u & G_D \end{bmatrix} = G_* + \Delta G$.

Proof: See Appendix II in [17].

V. Convergence to a Subspace

In many cases, and especially in quantum information systems, e.g. [2], it is desirable to have all state trajectories converge to appropriate well-behaved subspaces. In the Quantum systems situation, the appropriate subspace is a "decoherence-free subspace" as described in [20]-[22]. These subspaces are finite-dimensional, Hamiltonian - invariant subspaces of the Schrodinger partial differential equation representing the quantum dynamics of the information system. In such a subspace S the decoherence effects of the environment are removed, i.e. the Schrodinger dynamic group is unitary on S and thus preserves the energy in all states in the decoherence-free subspace. Therefore, within S, quantum information can be handled with quantum gates that do not lose information through decoherence.

In this section, we will deal with the general problem of adaptively controlling the states of a linear infinite-dimensional system to converge to a prescribed subspace. The prescribed subspace S will be an A-invariant subspace of the state space X in (1) with $\dim S \equiv N < \infty$. Consequently S is closed and $X = S \oplus S^\perp$.

Let $P_N \equiv$ orthogonal projection onto S along $S^\perp$, with $P_R \equiv I - P_N$ the complementary bounded projection. We define Convergence to a Subspace S of a trajectory $x(t)$ as $d(x(t), S) \xrightarrow[t\to\infty]{} 0$,

or equivalently $P_R x(t) \equiv (I - P_N)x(t) \xrightarrow[t\to\infty]{} 0.$

So, for the above pair of projections (P_N, P_R), we have $P_N P_R = 0$ and $AP_N = P_N A$, $AP_R = P_R A$, because S is A-invariant, and the linear infinite dimensional system (1) decomposes into

$$\begin{cases} \dfrac{\partial P_N x}{\partial t} = P_N \dfrac{\partial x}{\partial t} = (\underbrace{P_N A P_N}_{A_N}) P_N x + (\underbrace{P_N A P_R}_{A_{NR}}) P_R x + (\underbrace{P_N B}_{B_N}) u \\ \dfrac{\partial P_R x}{\partial t} = P_R \dfrac{\partial x}{\partial t} = (\underbrace{P_R A P_N}_{A_{RN}}) P_N x + (\underbrace{P_R A P_R}_{A_R}) P_R x + (\underbrace{P_R B}_{B_R}) u \\ y = (\underbrace{C P_N}_{C_N}) P_N x + (\underbrace{C P_R}_{C_R}) P_R x \end{cases} \quad (14)$$

Let $x_N \equiv P_N x$ and $x_R \equiv P_R x$,

which yields to $\begin{cases} \dfrac{\partial x_N}{\partial t} = A_N x_N + A_{NR} x_R + B_N u \\ \dfrac{\partial x_R}{\partial t} = A_{RN} x_N + A_R x_R + B_R u \\ y = C_N x_N + C_R x_R \end{cases}$.

Since S is A-invariant, we have $A_{NR} = P_N A P_R = P_N P_R A = 0$ and similarly $A_{RN} = 0$.

$$\Rightarrow \begin{cases} \dfrac{\partial x_N}{\partial t} = A_N x_N + B_N u \\ \dfrac{\partial x_R}{\partial t} = A_R x_R + B_R u \\ y = C_N x_N + C_R x_R \end{cases}$$

By choosing actuators $\{b_1, ..., b_M\}$ and sensors $\{c_1, ..., c_M\}$ in $S^\perp$, we can have $B_N = P_N B = 0$ and $C_N = CP_N = 0$.

$$\Rightarrow \begin{cases} \dfrac{\partial x_N}{\partial t} = A_N x_N \\ \dfrac{\partial x_R}{\partial t} = A_R x_R + B_R u \\ y = C_R x_R \end{cases}$$

Theorem 4: If (A_R, B_R, C_R) is ASD (i.e. $C_R B_R > 0$ and $C_R(sI - A_R)^{-1} B_R$ is minimum phase, then the direct adaptive controller (4a) and (4b) will produce $\|x_R = P_R x\| \xrightarrow[t\to\infty]{} 0$ (convergence to the subspace S) $\forall x_0 \in D(A)$ with bounded adaptive gain $G(t)$ (and will mitigate persistent disturbances if they are present in the (A_R, B_R, C_R) subsystem).

VI. Application: Adaptive Control of Hamiltonian Quantum Systems

In general, the dynamics of quantum systems are described by the Schrodinger wave equation on a complex Hilbert space [18]-[19]. We will apply the above direct adaptive controller on the following single-input/single-output Cauchy problem which represents *a feedback-controlled quantum system* with one control actuator and one sensor:

$$\begin{cases} \dfrac{\partial x}{\partial t} = Ax + b(u + u_D), x(0) \equiv x_0 \in D(A) \\ y = (c, x), \text{ with } b = c \in D(A) \end{cases} \tag{15}$$

where *A is the Hamiltonian operator for the quantum system* which is self-adjoint, has compact resolvent, and generates a C_0 semigroup.

From the compact resolvent property, we have that every state in the Hilbert space can be represented as $x = \sum_{k=1}^{\infty} c_k \phi_k$, where ϕ_k are the orthonormal eigenstates of A and are the so-called pure states of the system. Thus x is in general a *mixed state* where $\sum_{k=1}^{\infty} |c_k|^2 = 1$, and the $|c_k|^2$'s are the probabilities that the measured state is the pure state ϕ_k.

Consequently, there exists G_* such that $A_c \equiv A + BG_*C$ satisfies $\operatorname{Re}\lambda_k \le -\mu < 0, \forall \lambda_k \in \sigma_p(A_c)$, which implies that

$$\begin{aligned} \operatorname{Re}(A_c x, x) &\equiv \frac{1}{2}[(A_c x, x) + \overline{(A_c x, x)}] = \frac{1}{2}[(A_c x, x) + (x, A_c x)] \\ &= -(Qx, x) \le -\mu \|x\|^2 ; x \in D(A_c) \end{aligned}.$$

Also, since $b = c$ we have $C^* = B$. Therefore, we have that (A, B, C) is ASD with $P = I$.

From $\operatorname{Re}(A_c x, x) \le -\mu \|x\|^2, \forall x \in D(A)$ we clearly have $\operatorname{Re}(A_c x, x)$ bounded on bounded sets of $x \in D(A)$.

For this application we will *assume the disturbances are sinusoidal with frequency 1 rad/sec* (but this is not a restriction as long as φ_D is bounded:

$$\begin{cases} u_D = \begin{bmatrix} 1 & 0 \end{bmatrix} z_D \\ \dot{z}_D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z_D \end{cases}$$

implies that $F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \theta_D = \begin{bmatrix} 1 & 0 \end{bmatrix}; \varphi_D \equiv \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, and implies that $u = G_e y + G_D \varphi_D$ with $\begin{cases} \dot{G}_e = -yy^* \gamma_e \\ \dot{G}_D = -y\varphi_D^* \gamma_D \end{cases}$.

So, since $B = \Gamma$, there is a gain $S_2 = -\theta$ such that $BS_2 + \Gamma\theta = B(-\theta + \theta) = 0$, which implies that $S_1 = 0$, and this is the special case of (7). Finally, $E = 0$, and the eigenvalues of F are $\pm j$, but the zeros of *(A, B, C)* are real; so the matching conditions are satisfied and ideal trajectories exist. Therefore, we satisfy the hypothesis of Theo. 3 and we have, via the direct adaptive controller, state regulation $x \xrightarrow[t\to\infty]{} 0$ and output regulation $y \xrightarrow[t\to\infty]{} 0$ with bounded adaptive gains $G \equiv \begin{bmatrix} G_e & G_D \end{bmatrix}$ in the presence of sinusoidal persistent disturbances.

We note that quantum control would more likely be done with a master equation involving density operators rather than the usual Schrodinger equation in (15); also, the interaction with the environment would be modelled by an appropriate Lindblad operator. But the above gives a start at a framework for adaptive quantum control.

VII. Conclusions

In Theorem 1, we showed conditions under which ideal trajectories exist for a linear infinite-dimensional system to be capable of rejecting a persistent disturbance in the output of the plant. In Theorem 3 we used an extension of Barbalat-Lyapunov result for linear dynamic systems on infinite-dimensional Hilbert spaces under the hypothesis of almost strict dissipativity for infinite dimensional systems, to show that direct adaptive control can regulate the state and the output of a linear infinite-dimensional system in the presence of persistent disturbances without using any kind of state or parameter estimation. In Theorem4, we began the development of adaptive control causing a quantum system to converge to a decoherence -free subspace where quantum error-correction can operate. The control of a simple quantum system is described by a general Schrodinger wave equation with external disturbances using a single actuator and sensor and direct adaptive output feedback.

These results give a basic framework for direct adaptive control of quantum systems. They are meant as a beginning for the use of adaptive control in this context. They show that adaptive control does not require deep knowledge of specific properties or parameters of the system to accomplish decoherence reduction. But there are still many technical issues to overcome.

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