Non-Linear Mathematical Models based on Analytical Multiplicative-Additive Transformations Approximating Experimental Data

Rudolf A. Neydorf
Department of Software Computer Technology and Automated Systems
Don State Technical University
Rostov-on-Don, Russia
e-mail: rudolf.neydorf.44@mail.ru

Anna R. Neydorf
Department of Aquaculture technology
Don State Technical University
Rostov-on-Don, Russia
e-mail: neydan@yandex.ru

Dean Vučinić
Vesalius College (VeCo)
Vrije Universiteit Brussel (VUB)
Brussel, Belgium
e-mail: dean.vucinic@vub.be

Anatoly R. Gaiduk
Department of Control Systems
Southern Federal University
Taganrog, Rostov region, Russia
e-mail: gaiduk_2003@mail.ru

Nikita V. Kudinov
Department of Software Computer Technology and Automated Systems
Don State Technical University, Rostov-on-Don, Russia
e-mail: kudinov_nikita@mail.ru

***Abstract*—The paper presents the creation of mathematical models from experimental data, which are nonlinear and multidimensional. Such approach is required in many computer-based simulations for a variety of technical objects. It is well known that modern mathematical methods are not able to define, at the same time, both the needed numerical accuracy and the analytical properties. The presented approximation method for non-linear experimental data is a highly accurate mathematical model which contains interesting analytical properties. The method correctness has been validated analytically and experimentally for 1-dimensional and 2-dimensional objects. The method accuracy is based on the data "piecewise" approximation, i.e. their local fragmentation. However, in contrast to the "piecewise" approximation, the numerical model fragments are combined by multiplicative-additive transformation, and not by fulfilling coordinate-logical conditions. Therefore, such numerical model has analytical properties, which are used in the mathematical transformations, as multiplicative transformation of the created analytic functions, called "cut out" functions. These functions are locally approximating the data fragments and have analytical properties, which enable them to be added, when the combined analytical model is defined. The method implementation consists of the following successive and relatively independent stages: (1) splitting the data array into fragments, (2) their polynomial approximation, (3) the multiplicative transformation of fragments and (4) their additive combination into only one analytical function. The proposed method is appropriate for the experimental mathematical modeling of complex non-linear objects, in particular, for their use in the physical and technical simulation processes. The authors envisage that this proposed method would be especially useful for the mathematical modeling of the physics occurring in turbulent flows.**

Keywords*—*nonlinear multidimensional mathematical models; experimental data approximation; multiplicative analytical transformations.

I. INTRODUCTION

The creation of Mathematical Models (MM) for various simulations, objects, systems, etc., involves acquiring their experimental raw-point data having a certain structure. Usually, the experimental input data is acquired on a regular basis, and often the principle of so-called variation by coordinates is used. The processed variables can have duplicate values, which allow the use of matrix algebra and matrix-vector data representation, as multidimensional tables and multidimensional matrices. Difficulties of formally presenting the data arise, but this can be overcome by applying the proposed mathematical methods. However, the complexity increases when the data do not have a smooth, but a fragmented structure. The term "fragmented structure" is related to the point data arrangement, when there are clearly separated fragments with significantly different slopes along the lines of their interfacing boundaries. The tabular or matrix representation does not always allow evaluating the fragments data structure nature. However, this property is clearly shown in Fig. 1.

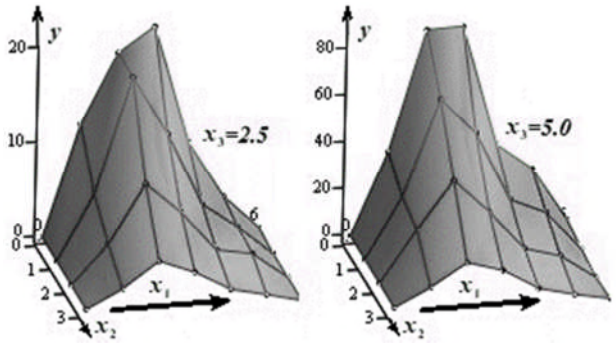

Figure 1. An example of 3-dimensional dependence $y = f^{(3)}(x_1, x_2, x_3)$ with fragment structure

The methods used can describe the "kink" with the desired accuracy. The most simple and effective approach is to apply the piecewise approximation [1] - [3] with any

 ISBN: 978-1-61208-732-0

required accuracy. However, the resulting MM has discontinuities in its derivatives along the fragments interfacing boundary lines, and thus excludes analytical transformations. The fragments spline approximation [4] - [6] disables the use of analytical MM transformations. In addition to the "fragment-oriented" approaches, there are methods aimed to build a single analytical MM, such as regression analysis [7] - [10], polynomial expansions [11] [12], methods of radial basis functions [13] [14] and others. It is important to note that they do not provide accurate data approximation for “kinks”, discontinuities and finding multiple extremes.

In order to eliminate the existing approximation methods drawbacks, as mentioned above, the Cut-Glue-Approximation (CGA) method was developed, having the first significant results published in [15] [16]. The CGA method combines the MM analytical properties and their quantitative accuracy. The MM approximation is based on processing the Experimental Data (ED) points arrays acquired for complex non-linear and multi-dimensional objects.

One important CGA method feature is the description of the nonlinear piecewise dependence by a single Analytical Function (AF), which has an additive structure consisting of multiplicative components. Multiplying the 2 CGA structure-forming functions: (1) the local approximating fragment (LAF) function of the ED array, and (2) a Multiplicative Isolating Function (MIF) applied to LAF, creates each component. The analytical multiplication of LAF by MIF “cuts out” the LAF section along the boundaries of the ED Fragment (EDF). This leads to a new AF, called Interval-Isolated Function (IIF), having the following unique properties:

1. The IIF fragment boundaries values coincide with LAF;
2. The IIF values beyond the EDF boundaries are almost zero;
3. IIF retains the analytical property [16].

These 3 enumerated IIF properties determine the fourth property: the possibility of IIF addition obtaining only one United AF (UAF), which approximates the whole piecewise dependences and fulfills the imposed accuracy for all IIF-s. The parametric tuning of the coefficients is enabled through the MIF arguments and they allow adjusting the overall approximation accuracy [15] [16].

In [15], the CGA method base was developed and applied in solving the 1-dimensional case. In [16], this method is generalized to the 2-dimensional case. However, it is often necessary to approximate multidimensional dependencies [1] [4] [9] [10]. For example, the strength properties of the composite materials depend on both the concentrations of the components and the technological parameters of their production [17]. For example, the positioning of the robot multilink working body is described by essentially non-linear and multidimensional dependencies [18] [19]. In some medical applications, the positions of detected organs, modeled as mathematical objects, may have discontinuous derivatives. Finally, the solution related to controlling nonlinear objects, by obtaining their MM in any form, is not enough to find practical solutions. What is required is that such MM, with their AFs, have the possibility to find an analytical solution by the synthesis of these objects control laws [20] - [23].

Thus, the follow up presented solution to such problem is applying the CGA method to essentially nonlinear problems of high dimensionality; this is an extremely relevant and promising approach [24] [25].

II. PROBLEM FORMULATION

The CGA method is applied to create MMs of arbitrary objects dimension, having nonlinear dependencies [15] [16] [24] [25] for the acquired experimental data sets. In addition, the paper addresses the generic CGA methodology, which is unique and does not have any similar analogs in the modern approximation theory, as MIF can be constructed for any dimension n of the model under consideration, which ensures that the construction of any n-dimensional IIF, as well as combining them into a single analytic differentiable UAF function, is made possible to describe any nonlinear dependence, which, in addition, might have discontinuous derivatives. It is also important to mention that EDFs have common boundaries taking into account any factor space dimension, which ensures the full coverage of the definition domain, for approximating dependence without affecting the approximation accuracy. Finally, the multidimensional IIFs are assembled into UAF (object’s MM) by applying the algebraic summation to combine them.

III. THE CGA METHOD PROSPECTS

The ED object modeling, when applying CGA, is implemented in 4 stages, each applying various mathematical methods and using significantly different information technologies. Currently, the Russian Foundation for Basic Research supports the CGA method development. All its 4 development stages are carried out independently, and the future plan is to supplement the developed algorithms with feedback loops. The envisaged feedbacks are representing the corrective actions for the previously developed CGA stages. Such general approximation algorithm structure should improve their quality and the applicability to model MM for the experimental data. In this paper the applied CGA version has independent stages divided into 2 blocks. The first preparatory block contains the first and second stages of the ED preprocessing to approximate fragments. The second modeling block contains the third and fourth stage, which are responsible for assembling fragments into a single MM of an object modeling the acquired experimental data.

The first CGA preparatory stage is pre-partitioning the experimental data into EDF. The second CGA preparatory stage is the applying the EDF created in the first stage. The processing result of the preparatory block is a set of LAF, fragments of the ED array. The third stage, which is part of the CGA modeling block, performs the conversion of LAF into their local fragments - IIF. In the fourth stage, all IIF are combined in UAF, which is the MM of the object.

 ISBN: 978-1-61208-732-0

The fragmentation process divides the ED into areas which can be well approximated. For each variable, its multidimensional dependencies are divided into intervals. This is already difficult for 2-dimensional problems, since the dependency curvature of any coordinate can vary significantly when other coordinates change. This property is well illustrated by in Fig.1. For 3 -dimensional dependencies, it is practically impossible to show the fragmentation graphically. In addition, the CGA method uses the regular ED coordinate grid with the rectangular EDF faces structure. Their curvature is approximated as nonlinear hyper-surfaces, in general, arbitrary oriented. Therefore, it is extremely difficult to define an effective distribution of hyper-parallelepipeds. With an increase in the modeled dependence dimension, the complexity is related to the distribution of its properties in the factor space, and, accordingly, the fragments complexity increases the manifold dimension. From above mentioned, it can be concluded that the automation of the ED fragmentation procedure of any number of dimensions, and not only for $n \geq 3$, is challenging. The software supporting the CGA method is needed, especially to process the dependencies of 3 or more arguments. It should be noted that in the CGA methodology, the fragmentation of the ED multidimensional arrays differs significantly from the remaining CGA stages, which mostly perform mathematical analysis. Therefore, the CGA fragmentation can be considered as a separate task and it is not discussed in this paper.

In accordance with the CGA method paradigm [14] [15] [23] [24], its second stage performs the approximation of each *k*-th fragment, selected within the *n*-th dimension experimental data array. The fragments are approximated by explicit analytical LAFs defined with *n*-independent arguments, and forming the vector

$$\boldsymbol{x} = (x_1, \ldots x_n)^T, \quad (3)$$

which for each *k*-th is

$$\boldsymbol{x_k} = (x_{1k}, \ldots x_{nk})^T. \quad (4)$$

The functions of the vector argument x_k approximates the *k*-th fragment, which is denoted by *n*-LAF, and mathematically described by the expression

$$y_{jk} = \varphi_{jk}(\boldsymbol{x_k}), \quad (5)$$

The analysis of the results from [4-9] shows that constructing analytical functions to approximate experimental data having fairly smooth fragments of nonlinear objects should be carried out with a well-developed Classical Regression Analysis (CRA) [12] – [16]. When CRA is applied to the polynomial functions, the regression coefficients obtained from the Least Squares Method (LSM) can effectively optimized their parametric structure variation. In addition, Power Polynomials (PP) are extremely convenient to order the variation of their structure by including/excluding their respective polynomial members. This allows finding the appropriate accuracy by describing the MM for each EDF, with an effective parametric (in accuracy) and structural (MM complexity) optimization of the EDF approximation. The related implementation has structural regularity, which enables creating the Complete PP (CPP) with an arbitrary dimension *n* for the function y_{jk}, and approximation degree *m,* applying the following convenient recursive scheme:

$$y(\boldsymbol{x}) = b_0 + \Sigma_i b_i \cdot x_i + \Sigma_{ij} b_{ij} \cdot x_i \cdot x_j + \Sigma_{ijk} b_{ijk} \cdot x_i \cdot x_j \cdot x_k + \ldots \quad (6)$$

where $b_{ijk\ldots}$ are multiplicative compositions coefficients defining variants of polynomial members (containing 1 to *m* multiplied members), while the composition form «$x_i \cdot x_j \cdot x_k \ldots$» may also include the power functions of independent variables, up to one variable in the degree, which corresponds to the PP member order (for example, b222 · (x2) 3); the number of forms that define such polynomial structure is *n*.

To simplify and generalize the algorithm for calculating the optimal accuracy of the EDF approximation, by using such PP, a pseudo-extension of the polynomial dimension in accordance to the number of pseudo-independent variables x_p is done. In this case, the nonlinear expression (6) turns into a pseudo-linear polynomial of the following form:

$$y(\boldsymbol{x}) = b_0 + \Sigma_n b_i \cdot x_i + \Sigma_{(n+1),N} b_p \cdot x_p \quad (7)$$

where $p=n+1, N$; N-n is the number of possible multiplicative combinations x_i, $i=\epsilon(1, n)$ from 2 to n combination members; x_p - pseudo-arguments.

In the case of non-degeneracy of the matrix $(X^T X)$, the vector $\boldsymbol{b}$ coefficients of the pseudo-linear polynomial (7) are optimized by the LSM criteria using the following universal matrix formula:

$$\boldsymbol{b} = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{Y}, \quad (8)$$

where $\boldsymbol{Y}$ is the output vector, and $\boldsymbol{X}$ is the input matrix under study, consisting of columns of independent variables $\boldsymbol{x_j}$, the rows are the values of these variables $\boldsymbol{x_{ij}}$ in the *i*-th tries when acquiring the experimental data modeled to support the related MM object.

Since formula (8) is obtained from the Fermat's theorem conditions for the quadratic error function approximating the point data by a power polynomial (7) of any structure, the parametric optimum of the vector $\boldsymbol{b}$ is guaranteed. However, in the general case, each JV structure is characterized by its optimum value. As a result, the discrete structural and optimized approximation polynomial for the set of all possible structures (6) arises. Thus, the EDF approximation, based on the universal polynomial MM, is a convenient tool for its structural-parametric optimization.

The polynomial MM structural optimization and the need to search for its structural and parametrical optimal approximation variant is found interesting to simplify the model. Studies of the authors have shown that, often, having defined the complete polynomial is not the guarantee to have

 ISBN: 978-1-61208-732-0

the best approximation accuracy. This effect is due to the fact that the properties of some nonlinear terms contradict the nature of the approximated dependence. However, it is difficult to predict in advance, what are these polynomial terms, due to the peculiarities of the hyper surfaces curvature, which approximate the experimental data of each individual fragment.

The solution to such structural optimization problem requires separate explanation. 3 points require attention: 1) the methodology for solving a discrete optimization problem on a finite set of PP structures; 2) the technology for the formation and variation of these structures, which is associated with symbolic transformations, 3) the formalization and quantitative representation of the criteria for the PP complexity, which is heuristic in nature.

The first problem has two solutions. With a small number of dimensions for modeling the object input space and with a small order of the approximation EDF polynomial, these 2 factors together give a multitude of structures that can be considered in the foreseeable time, and the problem can be solved by the full enumeration method, i.e. being combinatorial. When the cumulative properties of the same factors make a similar NP-complete problem impossible to solve in a reasonable time, it is necessary to apply heuristic search engine optimization, and the best is to apply the modified evolutionary-genetic algorithm for this problem. Its de research and development is discussed in a separate publication.

The solution of the second problem is necessary both for the combinatorial and the search optimization approaches. For coding of the nonlinear members of the PP with the dimension of the ED array $n <= 9$, it is convenient to use structure (6) for PP. The index of the independent variable x indicates the presence of this variable in the multiplicative term of PP, and the degree to which this variable is raised is indicated by the number of repetitions of this index in the code in the PP term. Thus, the free term b, in the polynomial code description, is denoted by "0", the first order term $b_i x_i$ is denoted by "i" (for example, $b_3 x_3$ is "3"), the second order term $b_{ij} x_i x_j$ is denoted by "ij", the third order term $b_{ijj} x_i (x_j)^2$ is denoted by "ijj", etc. The variant itself of the coded structure CPP of the polynomial P to be evaluated is indicated by a sequence of numeric codes separated by spaces.

For example, a 4-order incomplete 3-dimensional incomplete PP is encoded as follows:

$$P = b + b_1 x_1 + b_3 x_3 + b_{22}(x_2)^2 + b_{112}(x_1)^2 x_2 + b_{233} x_2 (x_3)^2 +$$

$$+ b_{1222} x_1 (x_2)^3 + b_{2233}(x_2)^2 (x_3)^2 + b_{3333}(x_3)^4 \sim$$

$$\sim K_P = 0\ 1\ 3\ 22\ 112\ 233\ 1222\ 2233\ 3333 \qquad (9)$$

Their correspondence is shown in (9) by the equivalence sign "~".

Due to the fact that there is not yet a universal and reliable theory for estimating the complexity of such mathematical expressions, the structure of PP has 2 heuristically formulated criteria, which have been proposed and tested when creating the CGA method. The first one is based on the expert evaluation, in the hierarchy of which the difficulty of using a first-order term (see Table 1, column 2) is taken as a unit, while the rest are estimated as shares of it (see Table 1, column 3). The second criterion, although heuristic, has physical justification, since its scale is based on experimental estimates of the time resources spent on the calculation of each PP term (see Table 1, column 2) on the PC used by the authors of the configuration (see Table 1, column 4). Table 2 is a fragment of a general table that can be constructed for any dimension and order of the PP. The fragment is limited to the fourth order of the 3-dimensional PP. The heuristic approach to estimate the complexity applies an average hybrid complexity estimate by scaling the dimensional resource time estimate with a factor of 0.01 (Table 1, column 5).

TABLE I. QUANTITATIVE ESTIMATES OF THE POLYNOMIAL STRUCTURES COMPLEXITY

№№	The structure of a polynomial term	Expert evaluation	Resource evaluation	Hybrid evaluation
1	0	0.5	~150	2.0
2-4	1,2,3	1	~275	3.75
5-10	11,12,13,22,23,33	2	~315	5.15
11-20	111,112,… 233,333	3	~375	6.75
21-35	1111,1112,… 3333	4	~450	8.5
…	…	…	…	…

The use of all 3 criteria showed their consistency, but the actual adequacy of any of them is difficult to prove. The research in this direction is ongoing.

The structuring of ED in the form of the EDF boundary set and building the LAF for all EDF allows moving from the CGA preparatory stages to the experimental MM construction stages.

IV. MODELING OF NONLINEAR OBJECTS FOR EXPERIMENTAL DATA USING CGA METHOD

The final stage of the CGA method is implemented in 2 operations: 1) obtaining an IIF from LAF, and 2) obtaining an UAF from IIF. The first operation is named "Cut the Fragments of LAF" (CF), and the second - "Glue the Fragments of LAF" (GF) [12] - [14] [17]. These 2 operations for one EDF describe LAF of the form (1), and can be represented by the following expressions:

$$\text{CF}: \quad f_k(\boldsymbol{x}) = \varphi_k(\boldsymbol{x}) \prod_{ik=1\ldots nk} E^l(\boldsymbol{x}, x_{\text{L}ik}, x_{\text{R}ik}, \varepsilon_{\text{L}ik}, \varepsilon_{\text{R}ik}), \qquad (10)$$

$$\text{GF}: \quad F(\boldsymbol{x}) = \sum_{k=1\ldots K} f_k(\boldsymbol{x}), \qquad (11)$$

where $f_k(\boldsymbol{x})$ is the k-th IIF; $\varphi_k(\boldsymbol{x})$ is the k-th LAF; n is the co-ordinate dimensionality of the object; $E^l(*)$ is the i-th one-dimensional MIF for the k-th LAF with approximation tuning arguments: x_{Lik}, x_{Rik} –the margin values of the approximation ranges of the k-th EDF in the i-th variables; ε_{Lik}, ε_{Rik} - parameters of the steepness of the fronts of the k-th IIF with respect to the i-th variables; K is the number of EDF and their IIF, respectively; x is an independent vector argument of the modeling object.

In other words, to create IIF of AF that locally approximates EDF, it is required that this function is multiplicative isolated along each independent variable

coordinate xi with 1-dimensional MIF, denoted as 1-MIF. Their multiplication output forms n-MIF and supports their multidimensional LAF isolation. Thus, in expression (10) this multidimensional MIF is formulated as:

$$\boldsymbol{E}^{n}(\boldsymbol{x}, \boldsymbol{x}_L, \boldsymbol{x}_R, \boldsymbol{\varepsilon}_L, \boldsymbol{\varepsilon}_R) = \prod_{i=1\ldots n} E^{l}(\boldsymbol{x}, x_{\mathrm{L}i}, x_{\mathrm{R}i}, \varepsilon_{\mathrm{L}i}, \varepsilon_{\mathrm{R}i}), \qquad (12)$$

where $\boldsymbol{E}^{n}(*)$ is n-MIF; $\boldsymbol{x}_L$, $\boldsymbol{x}_R$, $\boldsymbol{\varepsilon}_L$, $\boldsymbol{\varepsilon}_R$ – its n–vectors settings. The multiplicatively forming $\boldsymbol{E}^{n}(*)$ 1-dimensional MIF $E^{l}(*)$ is the main system-forming artifact of the CGA method. In this regard, it is necessary to study the properties of 1-MIF, which determine the result of the CF operation.

The developed 1-MIF functions in the first stages, when creating the CGA method had the following structure:

$$E^{l}(\boldsymbol{x}, \boldsymbol{x}_B, \boldsymbol{\varepsilon}_B) = L(\boldsymbol{x}, x_L, \varepsilon_L)\, R(\boldsymbol{x}, x_R, \varepsilon_R)/B(\boldsymbol{x}, \boldsymbol{x}_B, \boldsymbol{\varepsilon}_B) = \qquad (13)$$

where
$$L(\boldsymbol{x}, x_L, \varepsilon_L) = x - x_L + [(x - x_L)^2 + (\varepsilon_L)^2]^{1/2};$$
$$L(\boldsymbol{x}, x_R, \varepsilon_R) = x_R - x + [(x_R - x)^2 + (\varepsilon_R)^2]^{1/2};$$
$$B(\boldsymbol{x}, \boldsymbol{x}_B, \boldsymbol{\varepsilon}_B) = 4\{[(x-x_{\mathrm{L}})^2+(\varepsilon_{\mathrm{L}})^2][(x_{\mathrm{R}}-x)^2+(\varepsilon_{\mathrm{R}})^2]\}^{1/2}.$$

Expression (13) contains 2 pairs of $\boldsymbol{x}_B = (x_L, x_R)^{\mathrm{T}}$ and $\boldsymbol{\varepsilon}_B = (\varepsilon_L, \varepsilon_R)^{\mathrm{T}}$ settings to implement the conditions of additional LAF approximation, when transforming it into an IIF using 1-MIF:

- Values of the left and right EDF interval boundaries x_L and x_R along the x_i axis;
- Coefficients ε_L and ε_R of the steepness of the left and right edges of a one-dimensional pulse, which is the 1-MIF, which cuts out the IIF fragment from LAF.

The interval parameters x_L and x_R of the function $E^{l}(*)$ in (13) are determined by the results of the execution of the ED fragmentation stage and strictly fulfill the conditions of boundaries identity of the neighboring EDFs:

$$\varepsilon_{\mathrm{L}i} \equiv x_{\mathrm{R}(i\text{-}1)};\ \ (\mathrm{L})\ \ x_{\mathrm{R}i} \equiv x_{\mathrm{L}(i+1)},\ \ (\mathrm{R}) \qquad (14)$$

which is due to the continuity requirement of the constructed MM.

More difficult is the problem of choosing the adequate values of the coefficients ε_L and ε_R. When moving from zero to a level equal to unity, on the left border of the x_L fragment, the slope of the pulse front and the smoothness of the transition to an asymptote parallel to the x_i axis are affected by the ε_L value. During the transition from the unit to the zero level on the right border of the x_R fragment, the same characteristics of the 1-dimensional pulse are affected by the setting ε_R.. Moreover, the smaller ε is, the closer is the function front to the threshold transition, and the narrower is the transition region between the vertical and horizontal asymptotes. For clarity, Fig. 2 shows a family of 1-MIF graphs for 3 values $\varepsilon_L = \varepsilon_R = \in \{0.01;\ 0.1;\ 0.3\}$, decoding the curves belonging.

The effect of ε on the accuracy of rectangular pulse reproduction is clearly distinguishable. It can be seen that already at ε<=0,01 in the range between x_L and x_R multiplication by 1-MIF of fragmentary LAF $\varphi_k(\boldsymbol{x})$ practically does not affects its values, fulfilling the accuracy sufficient for the engineering calculations. For ε<=0,001, the analytical "cutting out" of the LAF fragment does not practically differ from the conventionally logical one. In this case, also, all LAF values outside the boundaries x_L and x_R become little distinguishable from zero, which allows additive combining of the independent fragments without having noticeable distortion of their eigenvalues.

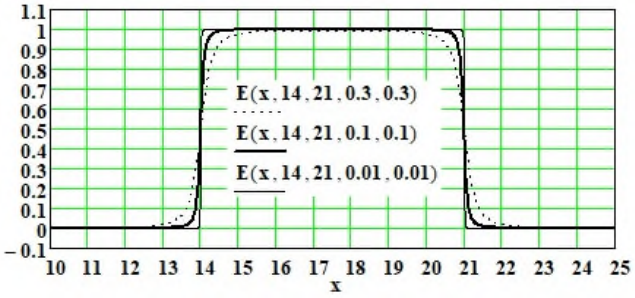

Fig. 2. Illustration of the approximation of the 1-MIF of the form (13) to an ideal rectangular pulse with a decrease in ε

An analytical study of the 1-MIF was carried out in [12]-[14]. These results showed that 1-MIF in the form of (13) have a number of properties that are useful for performing the analytical isolation of approximating functions sections. These properties are compactly summarized in Table II.

TABLE II. MAIN PROPERTIES OF 1-MIF, SUPPORTING THE MULTIPLICATIVE PROCESSING OF LAF

№	*Property Description*	*Values*
1	Function (13) is analytic, since decomposable in a Taylor series at any point in its domain of definition:	x∈]-∞, ∞[
2	Function (13) has a continuous range of values from zero to one:	E1(x)∈]0,1[
3	Function (13) has a single maximum x_{max}<1 at the point x_0, depending on $\boldsymbol{x}_B$, $\boldsymbol{\varepsilon}_B$:	E1extr=E1(x0)
4	The supremum $E^{l}_{\mathrm{supr}}(\varepsilon)$=1 for ε→0 and for any range $[x_L, x_R]$: x_L<x_R.	∀xL,xR&ε→0 → E1(ε)→1
5	Function (13) has zero infimum on the set x: $\lvert x\rvert\to\infty$	E1inf=0
6	The function (13) at small $\boldsymbol{\varepsilon}_B$ even near the boundary of the interval (for example, at a distance in 0.05 of the scale division) deviates very little from zero outside the interval and from the unit inside it.	ε=0.01→ ΔE1<0.01; ε=0.001→ ΔE1<0.0001
7	The function (13) with ε_L=ε_R is symmetric about the midpoint of the x_0 range of the approximated region, i.e. x_0 $E^{l}(x_0\text{-}\Delta x)$= $E^{l}(x_0+\Delta x)$	x0=(xL+xR)/2
8	At $\varepsilon_{\mathrm{L}}\neq\varepsilon_{\mathrm{R}}$, function (13) becomes asymmetric, which makes it possible to adapt the shape of the front of the IIF pulse to the boundary properties of neighboring EDF.	$E^{l}(x_0\text{-}\Delta x) \neq E^{l}(x_0+\Delta x)$
9	The values of function (13) at the pulse boundaries are within 0.25÷0.5 and depend only on the pulse width and on ε_B: $E^{l}{}_{B}(\Delta x,\varepsilon) = 0.25\cdot[\Delta x + (\Delta x^2 + \varepsilon_B)^{0.5}]/(\Delta x^2 + \varepsilon_B)^{0.5}$	E1B∈[0.25,0.5]

Analysis of the identified 1-MIF properties showed that all the formulated multiplicative paradigm tasks of the CGA method are performed.

First, the multiplicative processing preserves the analyticity property of the approximating function. That is,

IIF like LAF, a continuous analytic function (properties 1 and 2).

Second, 1-MIF has range values limits (infimum and supremum) 0 and 1. Multiplication of LAF and 1-MIF represents the cutouts of the LAF fragment, i.e. storing its values within the interval (with possible slight distortion - properties 3, 4). The same multiplicative operation isolates a fragment within its interval, making the LAF values outside the interval practically zero (property 5) for the entire ED definition domain.

Third, 1-MIF has the range limits (infimum and supremum) 0 and 1. When LAF and 1-MIF are multiplied, this provides a solution to the problem of cutting out the LAF fragment, i.e. storing its values within the interval (with possible slight distortion - properties 3, 4). The same multiplicative operation provides a solution to the problem of isolating a fragment within its interval, making the LAF values in the entire ED definition domain outside the interval practically zero (property 5).

Fourth, the magnitude of the distortion of the LAF values arising from its multiplicative processing manifests itself only near the fragment boundary, and is effectively regulated by the tuning arguments 1-MIF - εB = (εL, εR) T. In this case, the error can be made arbitrarily small (property 6). This property is illustrated in Fig. 3.

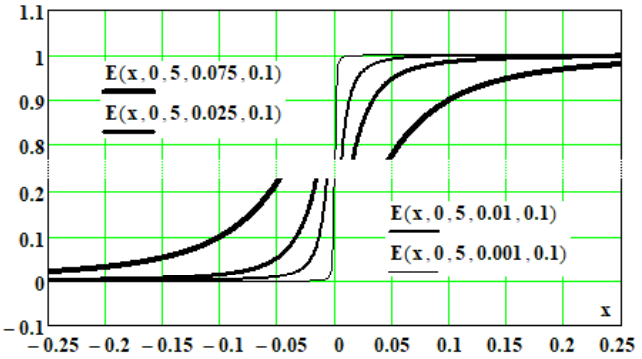

Fig. 3. Deviation dependence of 1-MIF values from 1 within the interval and from 0 beyond its boundaries

Five, the properties of symmetry and asymmetry, declared in clauses 7 and 8, offer ample opportunities for varying the 1-MIF settings to improve the quality of approximation of ED by the experimental MM as a whole. This is due to the fact that varying the value of εB can change the edge configuration of the IIF, reducing the error in joining the neighboring fragments. This factor is complemented by the special properties of the boundaries of 1-MIF, which are shown in paragraph 9.

The last of the 1-MIF properties considered is the variability property of the boundary settings. It plays a crucial role when used as the main tool of the CGA method. It makes it possible to formulate two important points. First, all the EDFs bordering on each other, formed at the first preparatory stage of the CGA method from the initial ED array, should have common borders (necessarily common, but not adjacent). Secondly, the tuning arguments of 1-MIF can be used for search optimization of the final error of the experimental MM obtained at the last stage of the CGA method.

The requirement postulated above for the fragmentation of ED at the EDF with an obligatory commonality of boundaries needs to be clarified in particular. Analysis of expression (13) for 1-MIF shows that the ordinate of crossing the border xB by its graph depends only on the pulse width $\Delta x = x_R - x_L$ and on ε_B, here ($B \in \{L, R\}$ depending on the boundary under study) . The formula for this dependence is given in the condition line 9. In Fig. 4, several curve graphics are plotted for several ε for this dependence. It is clearly seen that even with a sufficiently large ε = 0.2 ordinate, the intersection of the boundary abscissa is almost 0.5 already for Δx=2. When ε = 0.01, this is true in almost the entire range Δx=1. The value 0.5 is a supremum for $E^1(x_B, \boldsymbol{x}_B, \boldsymbol{\varepsilon}_B)$. Therefore, only additively combined boundaries of the EDF can reconstruct the values of the boundary experimental points with averaged accuracy of their description of LAF for neighboring fragments.

The considered rule is valid when implementing the GF-operation of the additive "gluing" IIF into a single UAF function, which is the experimental MM of the object under study. This is true not only for 1-dimensional ED. Studies have shown that $\boldsymbol{E}^n(\boldsymbol{x}, \boldsymbol{x}_L, \boldsymbol{x}_R, \boldsymbol{\varepsilon}_L, \boldsymbol{\varepsilon}_R)$, is formed by the product of n 1-MIF for all coordinates, according to (12). When multiplying by n-dimensional LAF, this n-dimensional MIF also forms n-dimensional IIF, which has common borders with neighboring IIF of various types. This is reflected in their boundary values.

In the 1-dimensional version, as shown above, 1-MIF has only 2 boundary points — the edges is a curve fragment, which gives a coefficient of ~0.5 for multiplicative processing.

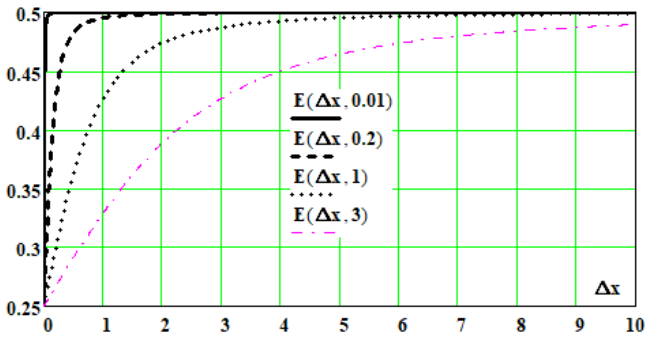

Fig. 4. Dependence of the 1-MIF borders properties on the pulse width and ε

This leads to the fact that the addition of two neighboring IIF $f_k(x)$ and $f_{k+1}(x)$, provided that $x_{Rk}=x_{Lk+1}$ and $E^1(x_{Rk},x_{Rk},\varepsilon_{Rk})=E^1(x_{Lk+1}, x_{Lk+1}, \varepsilon_{Lk+1})=0.5$, is determined by the expression

$$f_k(x_{Rk}) + f_k(x_{Lk+1}) =$$

$$= \varphi_\kappa(x_{Rk})\cdot E^1(x_{Rk},x_{Rk},\varepsilon_{Rk}) + \varphi_{\kappa+1}(x_{Lk+1})\cdot E^1(x_{Lk+1}, x_{Lk+1}, \varepsilon_{Lk+1}) =$$

$$= 0.5\cdot(\varphi_\kappa(x_{Rk}) + \varphi_{\kappa+1}(x_{Lk+1})) \qquad (15)$$

those will be equal to the average value of the output variable in LAF of these fragments in their common coordinate point.

As the dimension of the modeled dependencies increases, the number of different types of boundary sets increases linearly. So with n = 2 and the regular construction of the ED, which was declared at the beginning, its fragment will be a rectangle that is connected not with two segments along the x axis, but with 8 adjacent rectangles on the x_1-x_2 plane. At the same time, he forms 2 types of boundaries: straight line segments - 4 sides, and points - 4 vertices. At experimental points located on adjacent sides and common to 2 EDF, $E^2(x_{Bk},x_{Bk},\varepsilon_{Bk})$ will have a value of ~ 0.5, and at points that are vertices common to 4 EDF, the value 2-MIF will be equals ~ 0.25. This is easily verified by performing a mathematical analysis (13). At each such point four values are averaged to approximate this point for different EDF.

Table III shows the values of the MIF edge coefficients at the boundaries of various types for the dimensions of the objects being modeled up to and including 4. Very characteristic and useful for solving problems by the CGA method is the fact that the decreased coefficient of n-MIF for any of its boundary (boundary) point is always equal to the value close to the result of dividing one by the number of n-EDF bordering at that point.

The revealed phenomenon of MIF properties in the form of (12) allows, along with the main function of CF - the isolation of LAF values $\varphi_k(x_{Bk})$ along the fragment boundaries, to use CF to filter experimental errors and local approximation.

This allows to some extent reduce the overall methodological error of the ED approximation when performing a GF operation: the additive formation of the final UAF. The possibility of minimizing the error arises from the alternative trends in the dependence of the boundary values of 1-MIF on ε_{ij}. When ε_{ijL} and ε_{ijR} approach zero, the steepness of the isolated fragments fronts increases, i.e. improves the fragment isolation accuracy. The optimal values of the MIF settings - ε_{ij} can vary over a wide range of values, sometimes quite far from zero [12].

TABLE III. BOUNDARY VALUES N-MIF FOR EDF - PARALLELEPIPEDES UP TO 4 DIMENSIONS

n	1	2		3			4			
Border	Border point	Polygon side	Polygon vertex	Polyhedron face	Polyhedron edge	Polyhedron vertex	Polyhedron hyper surface	Polyhedron face	Polyhedron edge	Polyhedron vertex
Coefficient MIF	0.5	0.5	0.25	0.5	0.25	0.125	0.5	0.25	0.125	0.0625

However, the discussion on how the CGA method could be improved is too voluminous to be considered in this paper. Its follow up developments are envisaged in the coming future as extensions to the achieved results in this work.

V. CONCLUSION

The following 2 conclusions summarize the results achieved in this paper:

1. The presented CGA has a completely logical and has largely original structure, as follows:
   - Collection of experimental data (ED);
   - Fragmentation of ED to multiple EDFs;
   - Approximation of each EDF by a unique LAF;
   - Multiplicative transformation of each LAF using its unique function MIF with related IIF isolated in the argument space;
   - Algebraic summation of all IIFs forming UAF, which is MM of the studied object, created according to non-linear ED, but being, in this case, AF.
2. It was found that the CGA method characteristics are promising and motivate its application in solving the variety of engineering problems. The method has no analogues within the existing mathematical tools. It has practically unlimited application areas, especially adopted for modeling the non-linear n-dimensions dependencies. So far, the development and testing of the CGA method is validated for objects of the first and second order, which has been revealed and partially described in this paper, together with many of its promising developments aspects, to be undertaken in the future.

ACKNOWLEDGMENT

The research supporter by the Russian Foundation of Fundamental Research, project No. 18-08-01178/19 A.

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