A System of Pendulums on a Regular Polygon

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***Abstract*–We consider a dynamical system, which can be regarded as a transport model. A stochastic and deterministics versions of the model are investigated. The behaviour of the first version of the model is stochastic only at the beginning and over some time becomes a pure deterministic system. The second system comes to a steady state, which depends on the initial state. Considered models can be interpreted as cell automata**.

Keywords-dynamical system; transport models; synergy; cell automata.

I. Formulation of problem

We consider a mathematical model of a dynamical system. This model can be interpreted as a system of particles, which move in accordance with some rules. The movement of these particles is similar to movement of connected pendulums [1]. The model has also an equivalent interpretation. Namely, the support of movement can be considered as a closed sequence of contours. There are four vertices on each contour, as shown in Fig.1, and a particle, which occupies one of the vertices at each discrete time instant. Each contour has common vertices with two adjacent contours, as shown in Fig. 3. The model can be described as a Markov chain [7]. States of this chain correspond to configurations of particles.

We have obtained mathematical results that concern the behaviour of the system. The cases of small dimensions can be studied by exhaustion. Simulation is used in cases of greater dimensions.

The considered model is similar to a traffic model, which was introduced by K. Nagel and M. Schreckenberg and can be interpreted in terms of cellular automata [2]. Nagel and Schreckenberg investigated the movement on an one-dimensional lattice (straight line or circle).

In Section 1, the considered system is described. In Section 2, a formal description of particles movement rules is given. In Section 3, some propositions are formulated concerning a version of the model with a stochastic rule. In Section 4, some propositions are formulated concerning a version of the model with a deterministic rule.

A contour is considered, which contains four cells NWSE (North, West, South, East). A particle moves on the contour in accordance with rules formulated below. The rings can be joined at points (vertices) NWSE forming a network, as shown in Fig. 1 and Fig. 3.

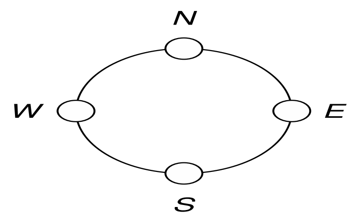

Fig. 1. Basis rings

Let us describe the rules of particles movement on the ring NWSE

(1) *Red state of particle A.* If at present time the cell C ahead of the particle A in the direction of movement is occupied by the particle B of another ring, then the particle A does not move.

(2) *Green state of particle A.* If at present time the cell ahead of the particle A in the direction of movement is vacant and not concurrent, then the particle A comes to C for one step.

(3) *Yellow state of particles A and B.* If at present time the same cell C is the next cell in the direction of movement for two particles A and B (no more particles can be as the network is plane), then, with probability $\alpha = \alpha(A)$, the particle A moves and the particle B does not move, and, with probability $\beta = 1 - \alpha$, the particle B moves and the particle A does not move.

The cell C is called concurrent.

Consider the following systems.

(a) We identify nodes N and S, W and E of the same ring, and we get an elementary "pendulum", Fig 2, $n = 1$.

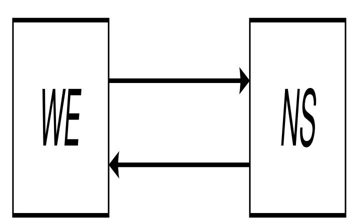

Fig. 2. A ring with joined opposite poles

(b) We consider also a "necklace", i.e., a closed system containing n rings, Fig. 3, $n > 1$.

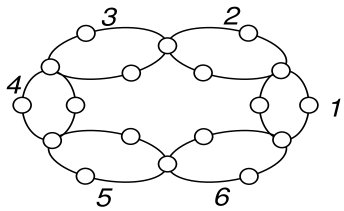

Fig. 3. Necklace

If we identify nodes N and S in each ring of "necklace", then we get a circular pendulum. Round nodes can be occupied successively by particles of the neighboring pendulums, and square nodes can be occupied only by their particle. A pendulum is a side of a regular polygon, as shown in Fig. 4.

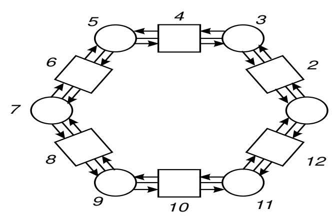

Fig. 4. Circular (hexagonal) pendulum

The main problem considered in the paper is to investigate the system behavior for different possible initial conditions:

(i) state *of synergy,* at which all particles move unimpeded;

(ii) state *of collapse,* at which all particles stop as they cannot move in accordance with the rules;

(iii) *spectrum of system velocities;* each particle, during time T, moves at T_1 steps and does not move at $T - T_1$ steps, or some particles stop and other move permanently.

The considered model is somewhat similar to models investigated in [3- 6], where cellular automata have been used for the local description of the traffic. The difference is that, in present paper, a network model has been introduced.

II. Circular α -n -pendulum

Consider a regular polygon with n vertices, as shown in Fig. 4. The cells are numbered from the vertex E, counter - clockwise $1, 2, 3, 4,$ as shown in Fig. 1. Even numbers correspond to main positions and odd numbers correspond to peripheral positions.

States of a particle, on the pendulum k at moment T, are denoted by

$x_k ='' (2k) + 1''$, if the particle occupies the cell $2k$ and moves in direction of $2k + 1$;

$x_k ='' (2k) - 1''$, if the particle occupies also the cell $2k$ but moves in the opposite direction;

$x_k =$ "$(2k + 1) - 1$", if the particle of the k-pendulum occupies at the right peripheral cell and moves back to the cell "$2k$";

$x_k =$" $(2k - 1) + 1$", symmetrically.

Suppose the cell 0 and the cell $2n$ are the same cell. The cell 1 and the cell $2n + 1$ are also the same cell. Suppose $x_1, x_2, x_3, \dots, x_m$ are the states of particles at present time. Then

(1) $(x_k) \neq (x_{k+1})\ \forall k,\ 1 \leq k \leq n$ as two particles cannot occupy the same cell;

(2) if $(x_k^*) = (x_{k+1}^*)$, then the cell $(2k + 1)$ is concurrent, and the probability of gain are α and $1 - \alpha$; i.e., with probability α realize

$$x_{k+1}^*(T) = x_{k+1}(T + 1),\ x_k(T + 1) = x_k(T),$$

or with $1 - \alpha$ respectively

$$x_k^*(T) = x_k(T + 1),\ x_{k+1}(T) = x_{k+1}(T).$$

(3) if $(x_k) = x_{k+1}^*$, then $x_{k+1}(T + 1) = x_{k+1}(T)$, i.e., the particle $k + 1$ does not move;

(4) if $x_k^*(T) = x_{k+1}^*(T)$, then $x_{k+1}(T+1) = x_{k+1}(T)$, the particle $k + 1$ does not move, i.e., $x_k(T + 1) = x_k(T)$.

III. Some results for α-n-pendulum

The following results have been found for the case $0 < \alpha < 1$.

3.1. *For all initial states, after a time interval with finite expectation, no concurrent cells occur.*

3.2. *For all initial states, the system comes to the state of synergy for a time interval with a finite expectation.*

3.3. *At the state of synergy, the same four states of the system are alternated with period 4.*

3.4. *If $n = 2$, then for all initial states $(T = 0)$ the system comes to the state of synergy no later than at time $T = 2$.*

3.5. *For any T, with non-zero probability, concurrent cells can still appear after time T.*

3.6. *Example one.* Let us fix $E = 1$, $N = 2$, $W = 3$, $S = 4$, as in Fig. 1. Let us consider one direct movement on necklace or equivalent pendulum with $n = 3$. Let (i_1, i_2, i_3) be a state of the system, where $i_j = 1$, if jth particle is at the right position, $i_j = 2$, if jth particle is at the middle position and moves to left; $i_j = 3$, if jth particle is at the left position, $i_j = 4$, if jth particle is at the middle position and moves to right.

Suppose the initial state is $(4, 4, 2)$. The following transitions can be realized

$$(4, 4, 2) \to (1, 4, 3) \to (2, 4, 4) \to (3, 1, 4) \to$$

$$\to (4, 2, 4) \to (4, 3, 1) \to (4, 4, 2).$$

The system has returned to the initial state after 6 steps, as shown in Fig. 5.

3.7. *Example two.* One more example. Consider a dynamical system of type shown in Fig. 3, with $n = 4$ and codirectional movement of particles. State of particle is denoted by letter R, or G, or Y.

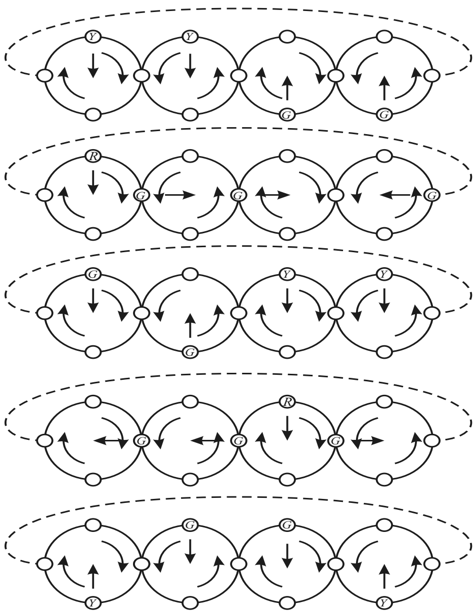

Fig. 5. Nondisappearing yellow color: step by step

Thus, some states with concurrent particles can be repeated with non-zero probability.

3.8. *Synergy effect.* Suppose $n = 3$, and all initial states are equiprobable. With probability 10/13 the system comes to the state of synergy than at time $k = 4$. With probability 3/13 the system comes to the state of synergy after time interval with expectation M

$$M \sim \frac{1}{1-\alpha},\ \alpha \to 1,$$

$$M \to \frac{1}{\alpha},\ \alpha \to 0.$$

In the last case, no finite number k exists such that with probability 1 the system comes to the state of synergy earlier than at time k.

3.9. *Digital synergy.* At the state of synergy, the configuration of particles is defined by position unique one.

IV. Right-priority n-pendulum

Suppose $\alpha = 1$ (analogously, $\alpha = 0$, left - priority). We follow Euler technology [10] of hypotheses burning.

4.1 *Qualitative property.* There are initial states such that the system comes to the state of synergy no later some fixed time, and there are such states that all particles move with same velocities that are less than 1 transition per time unit. For every initial condition, the average velocity of pendulum is greater than 0.5 transition per time unit.

4.2. Let us suppose $n = 2$. For all initial states $(k = 0)$, the system comes to the state of synergy no more than at time $k = 2$.

4.3. Let us suppose that $n = 3$ and all initial states are equiprobable. With probability 23/26 the system comes to the state of synergy no later than at time $k = 4$. With probability 3/26 the same 6 states alternate with period 6. In this case there are 4 transitions of every particle per a period, i.e., the velocity of every particle is equal to 2/3. The expectation of particles velocities is equal to 25/26.

4.4. Let us suppose that $n = 4$ and all initial states are equiprobable. With probability 75/97, the system comes to the synergy for a fixed time, and the same four states alternate with period 4. With probability 22/97, since some fixed time, the states of one of two sets alternate with period 16. In this case, there are 12 transitions of every particle per a period, and the velocity of particles equals 3/4. The expectation of particles velocities is equal to 183/194.

V. Conclusion and future work

We have considered a behaviour of a deterministic system. A stochastic and deterministics versions of the model are investigated.

A ”two-dimensional pendulum”, which is shown in Fig. 6, will be presented.

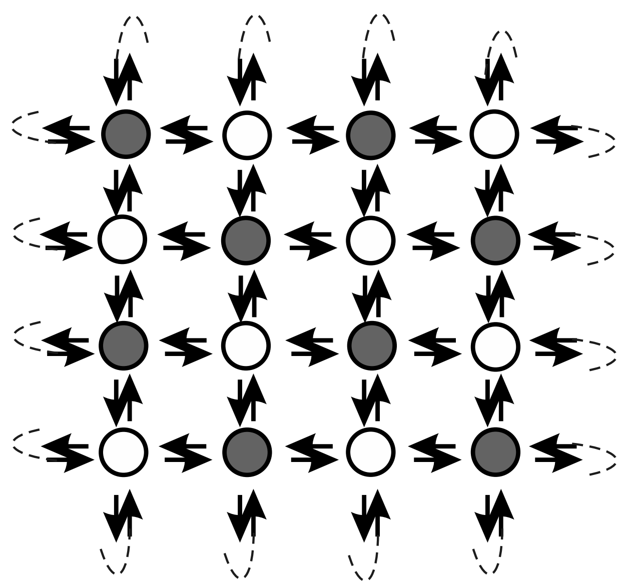

Fig. 6. Two-dimensional pendulum

Vertexes with even sum of row and column indexes, so called ”papa-vertexes”, contain particles, which move to neighbouring vertexes ”mama” according to some plan. One particular case is equivalent to dynamical model of flow on chainmail, as shown in Fig. 7.

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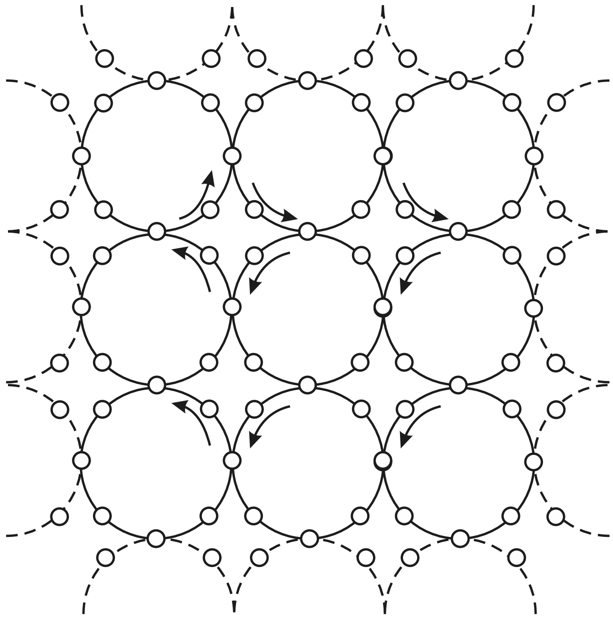

Fig. 7. Flow on chainmail

We also plan to discuss the connection with cellular play of Conway [9].

VI. Acknoledgments

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