A Control Framework for Direct Adaptive State and Input Matrix Estimation with Known Inputs for LTI Dynamic Systems

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***Abstract*—Equations Of Motion (EOM) can effectively describe the true physical dynamics within a set of assumptions and constraints. However, in many dynamic systems, the true physical system can deteriorate over time, leading to altered performance as the system ages and is utilized. These physical changes can be characterized as alterations in the constitutive contents and internal interactions. If these health changes are not accounted for in the EOM, discrepancies may emerge between the physical and the model responses. The proposed control scheme examines the case where the true system's plant and input matrix may experience a form of health change. The control scheme depends on knowing the true system's input and output state. The Lyapunov stability proof guarantees internal and external state error convergence to zero asymptotically if the true system experiences health changes within the assumptions and constraints of the proposed control scheme.**

Keywords-Adaptive; Control; Estimation; Plant; Input Matrix.

I. INTRODUCTION

Equations Of Motion (EOM) can describe the dynamics of the true physical system within a set of assumptions and constraints. However, dynamic systems may experience degradation over time or with usage. Failing to account for any deterioration resulting from changes in internal interactions or constitutive constants—such as mass, stiffness, and damping in mechanical systems—can lead to an inaccurate depiction of the true dynamics. Additionally, the potential decline in the system's actuator, which influences how inputs interact with the physical system, is often overlooked. As substantial portion of control problems involves regulating output error concerning a given input. Ignoring the health status changes in system dynamics or actuation can result in catastrophic failure if synthesized inputs do not adequately address these changes.

For traditional Luenberger or Kalman-like estimators to be practical, there has to be minimal uncertainty about the system [1][2]. Unlike Luenberger Estimators, Kalman-like filters are renowned for their ability to eliminate noise and stochastic variations resulting from sensor or process disturbances, under the assumption that the noise follows a Gaussian distribution centered around zero. However, neither type of estimator is capable of accommodating changes in the health status of the system dynamics or the input matrix.

The sensitivity of Luenberger and Kalman-like estimators to minimal uncertainty regarding system dynamics motivates the development of robustness techniques to address model uncertainty [3][4]. The control technique proposed here can manage both plant and input matrix uncertainties. More importantly, it can also accommodate significant changes in system health, as defined in the derivation. This work builds upon our earlier findings, which indicated that only the true-physical plant experiences a health status change, causing changes in dynamics and constitutive constants [5]. In 2022, *Griffith* developed a closed-loop approach for input matrix estimation [6]. This paper explores the scenario in which the plant and the input matrix experience a change in health.

The implemented control architecture was designed for a general system and can be applied to any system that meets the assumptions and constraints outlined in the proof. The proof relies on two primary stability criteria: Strict Positive Real (SPR) and Almost Strictly Dissipative (ASD). For a more formal definition and detailed explanation of SPR and ASD in the context of stability, please refer to [5][7]. Moreover, since none of the estimated states are fed back to the true system, the estimator can operate without risking harm to the true system. Additionally, the proposed control scheme can be utilized offline and online.

Following the introduction, this paper is divided into two main sections: III. Main Result and IV. Illustrative Example. The beginning of Section III offers a summary of the derivation process, presenting this paper's theorem and control diagram. Sub-Sections III-A and III-B provide the assumptions and constraints for both the true and reference systems while laying the foundation for updating the reference model. Sub-Section III-C defines the error states and their dynamics. In error dynamics, residual terms exist; therefore, error states cannot be guaranteed to converge to zero. To address this issue, the error dynamics are treated as energy-like terms. Then, an energy-like balance is constructed to remove residual terms, guaranteeing the error state to converge to zero globally as time approaches infinity. This process is detailed in Section III-D and III-E. Following the derivation, Section IV. Illustrative Example details the implementation of the derived control scheme. This example details a generic system where the error states converge to zero. The interaction tuning terms $\{\gamma_u, \gamma_u\}$ were left unadjusted in the example. However, tuning these terms can impact the time the error state converges.

II. Nomenclature

A	=	True Plant
A_m	=	Model Plant
ASD	=	Almost Strictly Dissipative
B	=	Input Matrix
B_m	=	Input Matrix Model
$\in$	=	Belongs
C	=	Output Matrix
$(\cdot)^\dagger$	=	Conjugate Transpose
e_x	=	Internal State Error
$\hat{e}_y$	=	External State Error
$\hat{}$	=	Estimate
$\forall$	=	For All
L_*	=	Fixed Correction Matrix
γ	=	Interaction Tuning Term
ΔL	=	Variance Matrix
PR	=	Positive Real
SPR	=	Strictly Positive Real
σ	=	Set of Eigenvalues
$\ni$	=	Such that
Re	=	Real
$\exists$	=	There Exists
u	=	Input
x	=	Internal State
y	=	External (Output) State

III. Main Result

Pertaining to the work being presented, the derived theorem and control laws, shown in Theorem 1 and Figure 1, are catered to minimizing the internal state error (e_x) to zero between true-physical system and reference model. This is achieved by accounting for discrepancies in the model plant (A_m) and input matrix (B_m), given a known input (u), output matrix (C), and external state (y). Uncertainty or variability in the model plant and input matrix means the convergence of the internal state error to zero cannot be guaranteed. As detailed in the derivation, to mitigate any variability, the error system is treated as an energy-like term. The aim is to dissipate all the energy of the error system, thereby ensuring the internal state error converges to zero as time approaches infinity, $e_x \xrightarrow[t\to\infty]{} 0$. To ensure error energy-like dissipation, the energy-like time rate of change for the error system is determined. Subsequently, residual energy-like time rate of change terms from any uncertainty are identified and countered. The remaining energy-like time rate of change term and the use of stability lemma, Barbalat-Lyapunov Lemma, ensures $e_x \xrightarrow[t\to\infty]{} 0$ asymptotically.

Theorem 1: Output Feedback on Reference Model for Adaptive Input Matrix, Plant, and State Estimation.

Consider the following state error system:

$$\begin{cases} \dot{e}_x = (A_m - KC)e_x + B_m\left(\Delta L_1 u + \Delta L_2 y\right) \\ \hat{e}_y = Ce_x = C(\hat{x} - x) \\ L_1 = \Delta L_1 + L_{1*} \\ L_2 = \Delta L_2 + L_{2*} \\ \dot{L}_1 = \Delta\dot{L}_1 = -e_y u^\dagger \gamma_u \\ \dot{L}_2 = \Delta\dot{L}_2 = -e_y y^\dagger \gamma_y \end{cases}, \quad (1)$$

where e_x is the estimated internal state error, $\hat{e}_y$ is the external estimated state error, $\{L_{1*}, L_{2*}\}$ are fixed-correction matrices, $\{\Delta L_1, \Delta L_2\}$ are the variability-uncertainty terms, K is a fixed gain, and $\{\gamma_u, \gamma_y\} > 0$ are interaction tuning terms. Given:

1) The triples of (A, B, C) and (A_m, B_m, C) are ASD and SPR respectively.
2) A model plant (A_m) must exist.
3) A model input matrix (B_m) must exist.
4) Output matrix (C) is known.
5) Allow $B \in \text{Sp}\{B_m L_{1*}\} \ni B \equiv B_m L_{1*}$.
6) Allow $A \in \text{Sp}\{A_m, B_m L_{2*} C\} \ni A = A_m + B_m L_{2*} C$.
7) The set of eigenvalues (σ) of the true and reference plant are stable (i.e. $\text{Re}(\sigma(A)) < 0$ & $\text{Re}(\sigma(A_m)) < 0$).

If conditions are met, then $\{e_x, \hat{e}_y\} \xrightarrow[t\to\infty]{} 0$ asymptotically. $\{\Delta L_1, \Delta L_2\}$ are guaranteed to be bounded; however, no guarantee of $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$. If $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$, then the dynamics of the true system or some energy equivalence have been numerically captured.

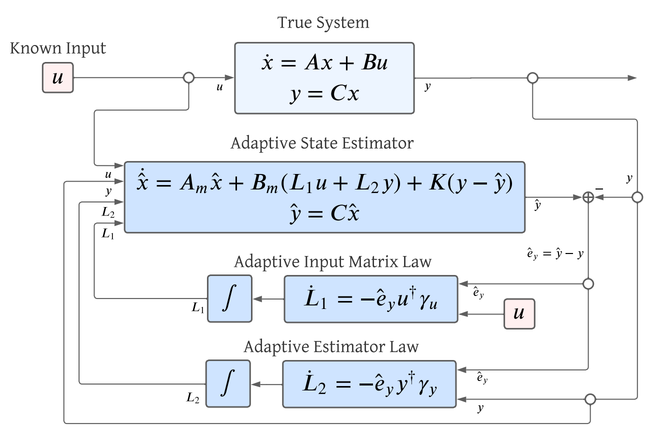

Figure 1. Control diagram for adaptive plant, input matrix, and state estimation given a known input (u), output matrix (C), and external state (y).

A. Defining True System Dynamics

Assume the dynamics of the true-physical system is linear time-invariant and therefore can be expressed in state-space form such that:

$$\text{True System} \begin{cases} \dot{x} = Ax + Bu \\ y = Cx. \end{cases} \quad (2)$$

Both the true system's plant (A), assumed to be stable (i.e. $\text{Re}(\sigma\{A\}) < 0$), and the input matrix (B) experience a health change caused by age or use, altering the constitutive constants and system dynamics. Output matrix (C) and external (output) state (y) are known. The input (u) can be any bounded-continuous waveform the user provides, possibly a known disturbance.

B. Overview of Updating the Reference Model

Subsequent sections will derive a control scheme and laws to minimize the error between the true and reference systems, (2) and (3), respectively. Note that both true and model systems match in dimension size.

$$\text{Reference Model} \begin{cases} \dot{x}_m = A_m x_m + B_m u \\ y_m = C x_m \end{cases} \tag{3}$$

To update the input matrix model (B_m), assume that B_m can be corrected via a input matrix fixed correction term (L_{1*}) such that:

$$B \equiv B_m L_{1*}. \tag{4}$$

The true plant is assumed to be decomposed into two components: an initial model (A_m) and plant matrix correction term $(B_m L_{2*} C)$ such that:

$$A \equiv A_m + B_m L_{2*} C. \tag{5}$$

Both (4) and (5) assumed decompositions are structured such that they can modified via an estimator. In the estimator, the initial input matrix and plant are updated via their respective correction term $\{L_1, L_2\}$:

$$L(t) = \Delta L + L_* \xrightarrow[t\to\infty]{} L(t) = L_*, \tag{6}$$

where ΔL is the variability-uncertainty term. If both variability term converges to zero, $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$, then the input matrix and true plant (or energy equivalent) have been numerically captured. For the control scheme to apply, the true and reference systems must be ASD and SPR, respectively.

C. Estimated State Error

Given that the true plant (A) and input matrix (B) experiences a health change caused by age or use and the internal state (x) is often blended into a linear combination or missing from the external state (y), an estimator can be created using the reference model:

$$\text{Estimator} \begin{cases} \dot{\hat{x}} = A_m \hat{x} + B_m (L_1 u + L_2 y) \\ \hat{y} = C \hat{x}. \end{cases} \tag{7}$$

To minimize the error between the true and estimated systems, consider the following error state equations:

$$\begin{cases} e_x = \hat{x} - x \\ \hat{e}_y = \hat{y} - y = C e_x. \end{cases} \tag{8}$$

To capture the internal state of the true system, the internal state error must converge to zero as time approaches infinity. To investigate the internal state error dynamics, take the time derivative of the internal state error and substitute (2) and (7):

$$\begin{aligned} \dot{e}_x &= \dot{\hat{x}} - \dot{x} \\ &= A_m \hat{x} + B_m (L_1 u + L_2 y) - (Ax + Bu). \end{aligned} \tag{9}$$

From (9), consider the difference between input matrices:

$$\begin{aligned} B_m(\underbrace{\Delta L_1 + L_{1*}}_{=L_1})u - \underbrace{B_m L_{1*}}_{=B} u &= B_m \underbrace{\Delta L_1 u}_{=w_u} \\ &= B_m w_u. \end{aligned} \tag{10}$$

Again, using (9) as a reference, consider the difference between the model and true plants, where $A \equiv A_m + B_m L_{2*} C$:

$$\begin{aligned} A_m \hat{x} + B_m(\underbrace{\Delta L_2 + L_{2*}}_{=L_2})y - Ax =& A_m e_x + B_m \underbrace{\Delta L_2 y}_{=w_y} \\ =& A_m e_x + B_m w_y. \end{aligned} \tag{11}$$

Therefore, the error system can be written as:

$$\begin{cases} \dot{e}_x = A_m e_x + B_m (w_u + w_y) \\ \hat{e}_y = C e_x. \end{cases} \tag{12}$$

Additionally, the estimator can be extended to use a fixed gain (K):

$$\begin{cases} \dot{\hat{x}} = A_m \hat{x} + B_m (L_1 u + L_2 y) + K(y - \hat{y}) \\ \hat{y} = C \hat{x}. \end{cases} \tag{13}$$

Resulting in the following error equation:

$$\begin{cases} \dot{e}_x = (\underbrace{A_m - KC}_{=A_c}) e_x + B_m (w_u + w_y) \\ \hat{e}_y = C e_x. \end{cases} \tag{14}$$

To use (14), find a fixed gain $(K) \ni \text{Re}(\sigma\{A_m - KC\}) < 0$.

Regardless of the estimator selected, the internal state error (e_x) can not be guaranteed to converge such that $e_x \xrightarrow[t\to\infty]{} 0$ due to the residual terms $\{w_u, w_y\}$ existing in the error equation. To adequately address these residual components, additional considerations are needed.

D. Lyapunov Stability for the Estimated State Error

Lyapunov stability analysis represents dynamic systems in terms of energy-like functions to describe the convergence of a particular or a set of states. For this case study, Lyapunov stability is used to guarantee the convergence of internal state error $(e_x) \ni e_x \xrightarrow[t\to\infty]{} 0$.

Given the state error equation as described in Eq.(12), consider the following energy-like Lyapunov equation with assumed real scalars:

$$V_e = \frac{1}{2} e_x^\dagger P_x e_x; P > 0, \tag{15}$$

where the $(\cdot)^\dagger$ is the conjugate transpose operator and where $P > 0$ represents a matrix P that is symmetric $(P_x = P_x^\dagger)$ and positive-definite $\text{Re}(\sigma\{P_x\}) > 0$.

To determine the energy-like time rate of change of V_e, take the time derivative of V_e and substitute (12) for the error dynamics:

$$\begin{aligned} 2\dot{V}_e =&\dot{e}_x^\dagger P_x e_x + e_x^\dagger P_x \dot{e}_x \\ =&(A_m e_x + B_m(w_u + w_y))^\dagger P_x e \\ &+ e_x^\dagger P_x(A_m e_x + B_m(w_u + w_y)) \\ =&e_x^\dagger (A_m^\dagger P_x + A_m P_x) e_x + 2\underbrace{e_x^\dagger P_x B_m(w_u + w_y)}_{=(B_m(w_u+w_y))^\dagger P_x e_x}. \end{aligned} \tag{16}$$

Modifying SPR stability condition for the reference model:

$$\begin{cases} A_m^\dagger P_x + P_x A_m = -Q_x \\ P_x B_m = C^\dagger \end{cases} ; Q_x > 0. \tag{17}$$

From here, the SPR condition can be applied to (16), resulting in:

$$\begin{aligned} 2V_e &= -e_x^\dagger Q_x e_x + 2\underbrace{e_x^\dagger C^\dagger}_{=\hat{e}_y^\dagger}(w_u + w_y) \\ &= -e_x^\dagger Q_x e_x + 2\hat{e}_y^\dagger w_u + 2\hat{e}_y^\dagger w_y \\ &= -e_x^\dagger Q_x e_x + 2\underbrace{(\hat{e}_y, w_u)}_{=(w_u,\hat{e}_y)} + 2\underbrace{(\hat{e}_y, w_y)}_{=(w_y,\hat{e}_y)}. \end{aligned} \tag{18}$$

By removing the residual terms $\{(\hat{e}_y.w_u), (\hat{e}_y, w_y)\}$ in (18), results in $V_e \leq 0$.

To counter the residual terms, consider the following energy-like functions:

$$V_u + V_y = \frac{1}{2}\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta L_1^\dagger) + \frac{1}{2}\text{tr}(\Delta L_2 \gamma_y^{-1} \Delta L_2^\dagger), \tag{19}$$

where $\{\gamma_u, \gamma_y\} > 0$. The energy-like time rate of change for $V_u + V_y$ follows:

$$\dot{V}_u + \dot{V}_y = \underbrace{\text{tr}(\Delta \dot{L}_1 \gamma_u^{-1} \Delta L_1^\dagger)}_{=\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta \dot{L}_1^\dagger)} + \underbrace{\text{tr}(\Delta \dot{L}_2 \gamma_y^{-1} \Delta L_2^\dagger)}_{=\text{tr}(\Delta L_2 \gamma_y^{-1} \Delta \dot{L}_2^\dagger)}. \tag{20}$$

A control law for the input matrix and plant variance time rate of change $\{\Delta \dot{L}_1, \Delta \dot{L}_2\}$ can be defined as the following:

$$\begin{cases} \Delta \dot{L}_1 = -e_y u^\dagger \gamma_u \\ \Delta \dot{L}_2 = -e_y y^\dagger \gamma_y. \end{cases} \tag{21}$$

Substituting (21) into (20):

$$\begin{aligned} \dot{V}_u + \dot{V}_y =&\text{tr}(\underbrace{-e_y u^\dagger \gamma_u}_{\Delta \dot{L}_1} \gamma_u^{-1} \Delta L_1^\dagger) \\ &+ \text{tr}(\underbrace{-e_y y^\dagger \gamma_y}_{\Delta \dot{L}_2} \gamma_y^{-1} \Delta L_2^\dagger) \\ =&-\text{tr}(e_y \underbrace{u^\dagger \Delta L_1^\dagger}_{=w_u^\dagger}) - \text{tr}(e_y \underbrace{y^\dagger \Delta L_2^\dagger}_{=w_y^\dagger}) \\ =&-\text{tr}(e_y w_u^\dagger) - \text{tr}(e_y w_y^\dagger) \\ =&-\text{tr}(w_u^\dagger e_y) - \text{tr}(w_y^\dagger e_y) \\ =&-w_u^\dagger e_y - w_y^\dagger e_y \\ =&-(w_u, e_y) - (w_y, e_y). \end{aligned} \tag{22}$$

For notation purposes, allow the following:

$$\begin{cases} V_{euy} = V_e + V_u + V_y \\ \dot{V}_{euy} = \dot{V}_e + \dot{V}_u + \dot{V}_y. \end{cases} \tag{23}$$

From here, the estimate state error closed-loop energy-like function can be written as:

$$\begin{aligned} V_{euy} =&\frac{1}{2}e_x^\dagger P_x e_x + \frac{1}{2}\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta L_1^\dagger) \\ &+ \frac{1}{2}\text{tr}(\Delta L_2 \gamma_y^{-1} \Delta L_2^\dagger). \end{aligned} \tag{24}$$

Therefore, the estimated state error closed-loop energy-like time rate of change can be written as:

$$\begin{aligned} \dot{V}_{euy} =&-\frac{1}{2}e_x^\dagger Q_x e_x + (w_u, \hat{e}_y) + (w_y, \hat{e}_y) \\ &-(w_u, \hat{e}_y) - (w_y, \hat{e}_y) \\ =&-\frac{1}{2}e_x^\dagger Q_x e_x \leq 0. \end{aligned} \tag{25}$$

Having $\dot{V}_{euy} \leq 0$ means that $\{e_x, \Delta L_1, \Delta L_2\}$ are guaranteed to be bounded. Due to $\dot{V}_{euy}$ negative-semi-definite nature, no additional information can be said about the error internal state (e_x) converging $\ni e_x \xrightarrow[t\to\infty]{} 0$.

E. Applying Barbalat-Lyapunov Lemma on $\dot{V}_{euy}$

To guarantee $e_x \xrightarrow[t\to\infty]{} 0$, consider Barbalat-Lyapunov Lemma - Given:

1) V is lower bounded.
2) $\dot{V}$ is negative-semi-definite.
3) $\dot{V}$ is uniformly continuous in time.

If all conditions are met, then $\dot{V} \xrightarrow[t\to\infty]{} 0$ according to [8].

The first two conditions of Barbalat-Lyapunov Lemma are satisfied with (24) and (25). The third condition, $\dot{V}_{euy}$ being uniformly continuous in time and can be satisfied by showing that $\ddot{V}_{euy}$ is bounded [8].

To prove $\ddot{V}_{euy}$ is bounded, consider W_{euy}:

$$W_{euy} \geq -\dot{V}_{euy} \geq 0. \tag{26}$$

Taking the time derivative of W_{euy} results in the following:

$$\begin{aligned} \dot{W}_{euy} =&2e_x^\dagger Q_x \dot{e}_x \\ =&2e_x^\dagger Q_x(A_m e_x + B_m(w_u + w_y)) \\ =&2e_x^\dagger Q_x(A_m e_x + B_m(\Delta L_1 u + \Delta L_2 y)). \end{aligned} \tag{27}$$

From (25), $\{e_x, \Delta L_1, \Delta L_2\}$ are bounded. Input (u) can be any bounded-continuous waveform. Following, the true plant is assumed stable ((i.e., $\text{Re}(\sigma\{A\}) < 0$); therefore, a bounded input will result in a bounded output (y) [9]. Combining all bounded results yields: $\dot{W}_{euy}$ is indeed bounded. Making $\ddot{V}_{euy}$ bounded.

Given that all the conditions of Barbalat-Lyapunov are satisfied, $\dot{V}_{euy}$ evolution in time can be expressed as:

$$\dot{V}_{euy} \xrightarrow[t\to\infty]{} 0. \tag{28}$$

Therefore, proves $e_x \xrightarrow[t\to\infty]{} 0$ is asymptotically guaranteed. However, regardless of Barbalat-Lyapunov being satisfied, Lyapunov stability results only guarantees $\{\Delta L_1, \Delta L_2\}$ to be bounded. If $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$ numerically, the true input matrix and plant or an energy equivalence have been captured. Additionally, without loss of generality, derived Lyapunov stability proof can be modified for the error system using fixed gain, Eq. (12).

Altogether, assuming the reference (A_m, B_m, C) and true (A, B, C) systems are SPR and ASD respectfully, such that the decomposition of the true input matrix (B) and plant (A) can be written as $B \equiv B_m L_{1*}$ and $A \equiv A_m + B_m L_{2*} C$. Then adaptive laws (Eq. (21)) and diagram (Figure 1) can be formulated such that the internal state error is guaranteed to converge zero asymptotically. Lyapunov stability proof only guarantees that $\{\Delta L_1, \Delta L_2\}$ will be bounded. However, if $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$, then the true input matrix and plant or energy equivalent have been numerically captured.

IV. Illustrative Example

The following is an illustrative example of applying Theorem 1 and the control diagram (Figure 1) on a general case study. Numerical values for (A_m, B_m, C) and (A, B, C) are derived and modified from [10].

A. State Space Representations for Reference and True Systems

Allow the reference model as defined in (3) have the following properties:

$$\begin{aligned} A_m &= \begin{bmatrix} -7 & 2 & 4 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{bmatrix}; \\ B_m &= \begin{bmatrix} 0 \\ .7 \\ 2 \end{bmatrix}; C = \begin{bmatrix} 0.5 & 0 & 1 \end{bmatrix}; x(0) = 0. \end{aligned} \tag{29}$$

To apply the control scheme as defined in Theorem 1 and show in Figure 1, allow the true system as defined by (2) have the following properties:

1) $B \in \text{Sp}\{B_m L_{1*}\} \ni B \equiv B L_{1*}$.
2) $A \in \text{Sp}\{A_m, B_m L_{2*} C\} \ni A \equiv A_m + B L_{2*} C$.

Assume the health change for the input matrix and plant can be described by $\{L_{1*}, L_{2*}\} \ni L_{1*} = 2$ and $L_{2*} = -5$. Therefore, the true system can be defined by the following:

$$\begin{aligned} A \equiv& A_m + B_m L_{2*} C = \begin{bmatrix} -7 & 2 & 4 \\ -3.75 & -1 & -1.5 \\ -7 & 2 & -11 \end{bmatrix}; \\ B \equiv& B_m L_{1*} = \begin{bmatrix} 0 \\ 1.4 \\ 4 \end{bmatrix}; C = \begin{bmatrix} 0.5 & 0 & 1 \end{bmatrix}; x(0) = 0. \end{aligned} \tag{30}$$

Recall that the constitutive constants of the true plant (A) and input matrix (B) are unknown. However, an initial estimate of the plant (A_m) and input matrix (B_m) exists.

When both the reference and true systems, as defined in (29) and (30), are given a unit step input, as shown in Figure 2, the differences in rise times and output response become evident. These differences can be further explained by examining the eigenvalues of the reference and true plants:

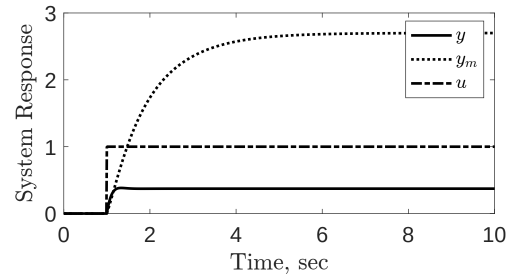

Figure 2. True (y) and reference model (y_m) output response given a unit step input (u).

$$\begin{aligned} \sigma(A_m) &= \{-1, -3, -5\} \\ \sigma(A) &\approx \{-2.28, -8.36 \pm i5.05\}. \end{aligned} \tag{31}$$

B. Defining the Known Input (u)

To implement the control scheme, a bounded and continuous input must be used. In practice, this input can be a known disturbance. For this example, allow the input be defined as:

$$u = 2 + \sin(2t). \tag{32}$$

C. Adaptive Estimation

In this section, the proposed control scheme detailed in Figure 1 is implemented with two cases: with and without the use of a fixed gain (K) term.

1) Adaptive Control Scheme without the use of Fixed Gain $(K = 0)$: The control scheme detailed in Figure 1 is implemented without using the fixed gain term $(K = 0)$ and $\{\gamma_u, \gamma_y\} = I$. As derived in the proof, Figure 3 demonstrates the convergence of the internal state, where $e_x \xrightarrow[t\to\infty]{} 0$. Given that the internal state error converges to zero, equivalently, the external state error convergences $\ni \hat{e}_y \xrightarrow[t\to\infty]{} 0$. Meaning that the estimated output $(\hat{y})$ converges to the true output (y).

Although the proof only guarantees that the adaptive variance will be bounded, numerically $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$. For this case study, the true input matrix and plant have been numerically captured, Figure 4 and Figure 5.

2) Adaptive Control Scheme with the use of Fixed Gain $(K \neq 0)$: The control scheme detailed in Figure 1 is implemented using the fixed gain term $(K \neq 0)$ and $\{\gamma_u, \gamma_y\} = 1$. The fixed gain term K was derived using a Linear Quadratic Regulator where $Q = I_3$ and $R = 1$. Similarly to the result of Section IV-C1, $e_x \xrightarrow[t\to\infty]{} 0$, shown in Figure 6. Again, since the internal state error converges to zero, the external error will converge to zero for the true and estimator systems. Moreover, as $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$ the true input matrix and plant are numerically captured in Figures 7 and 8.

There can be benefit of using a fixed gain term in the estimator, as the term can affect the time in which internal states and adaptive terms converge, compare Figure 4 and

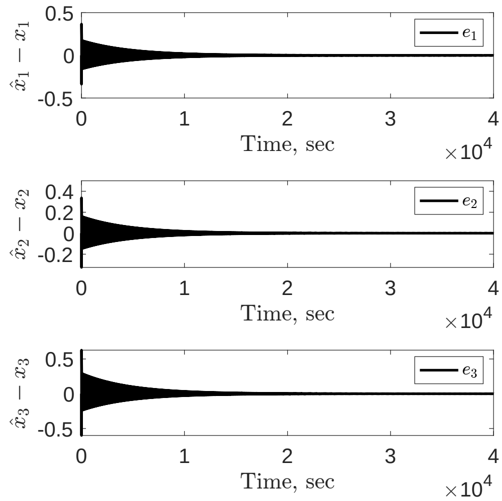

Figure 3. Internal state error converging to zero without the use of the fixed gain ($K = 0$).

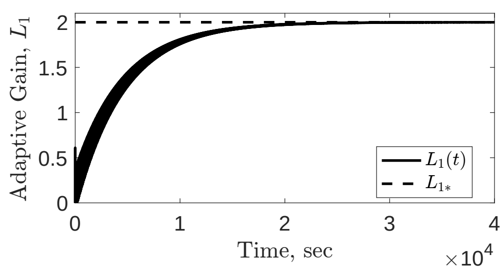

Figure 4. Input Matrix adaptive term $L_1(t)$ converging to L_{1*} without the use of the fixed gain ($K = 0$).

Figure 7. More crucially, both adaptive tuning terms $\{\gamma_u, \gamma_y\}$ can be adjusted to amply or dampen the effects of the adaptive controller, directly impacting the convergence of the error state. For this particular example, setting $\gamma_u = 1.3$ and $\gamma_y = 1.85$ reduces the time in which $e_x \xrightarrow[t\to\infty]{} 0$ and $L \xrightarrow[t\to\infty]{} L_*$ by order of magnitude faster than the depicted figures in this text. However, there are numerical limits for the tuning terms $\{\gamma_u, \gamma_y\}$. Making the adaptive controller too sensitive to changes may lead to divergent artifacts.

V. Conclusion

A physical system can experience wear and tear with use or age, altering performance. This paper examines the case where the true plant and the input matrix undergo a change in health, described as alterations in constitutive constants or internal interactions. If these health changes are not considered in the model, discrepancies in the true and model system dynamics can occur. This work proposes addressing the change in the true system's health by updating the model of the plant and input matrix according to their respective adaptive laws. If the assumptions and constraints of the proof are met, the adaptive laws will ensure that both internal and external state errors converge to zero asymptotically.

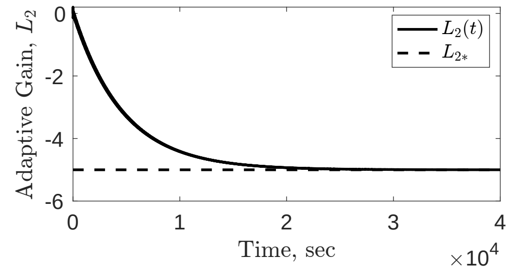

Figure 5. Plant correction adaptive term $L_2(t)$ converging to L_{2*} without the use of the fixed gain ($K = 0$).

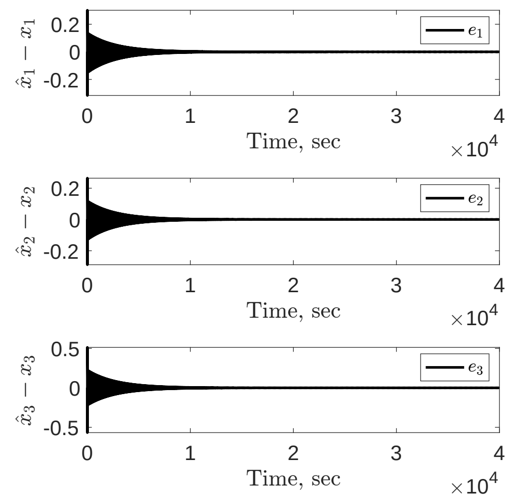

Figure 6. Internal state error converging to zero without the use of the fixed gain ($K \neq 0$).

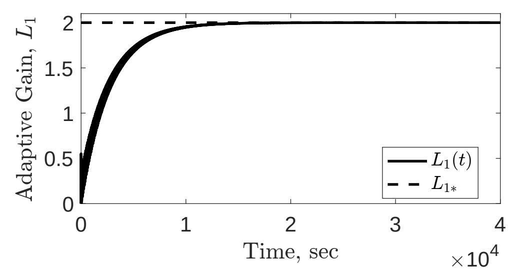

Figure 7. Input Matrix adaptive term $L_1(t)$ converging to L_{1*} with the use of the fixed gain ($K \neq 0$).

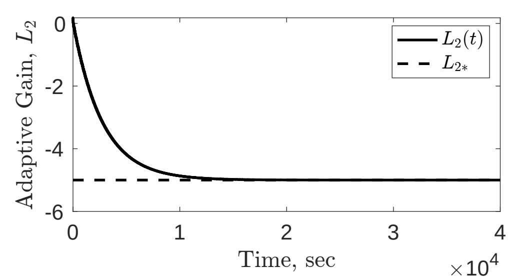

Figure 8. Plant correction adaptive term $L_2(t)$ converging to L_{2*} with the use of the fixed gain ($K \neq 0$).

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