

# Geometric Error Estimation

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**Abstract**—An essential prerequisite for the numerical finite element simulation of physical problems expressed in terms of PDEs is the construction of an adequate mesh of the domain. This first stage, which usually involves a fully automatic mesh generation method, is then followed by a computational step. One can show that the quality of the solution strongly depends on the shape quality of the mesh of the domain. At the second stage, the numerical solution obtained with the initial mesh is generally analyzed using an appropriate a posteriori error estimator which, based on the quality of the solution, indicates whether or not the solution is accurate. The quality of the solution is closely related to how well the mesh corresponds to the underlying physical phenomenon, which can be quantified by the element sizes of the mesh. An a posteriori error estimation based on the interpolation error depending on the Hessian of the solution seems to be well adapted to the purpose of adaptive meshing. In this paper, we propose a new interpolation error estimation based on the local deformation of the Cartesian surface representing the solution. This methodology is generally used in the context of surface meshing. In our example, the proposed methodology is applied to minimize the interpolation error on an image whose grey level is considered as being the solution.

**Keywords**—a posteriori error estimation; interpolation error; mesh adaptation; surface curvature.

## I. INTRODUCTION

Different kinds of estimators are available to a posteriori control the error made on a finite element solution [1]. Using such an estimator, it is possible to control the mesh by *h-adaptation* so that the corresponding solution of the PDE problem has a given accuracy. Some of these estimators are based on the interpolation error (and, in this sense, are purely geometric since they ignore the nature of the operator considered). This kind of estimators has been studied by many authors [2]–[5]. However, most of these studies lie on the fact that a parameter  $h$ , representing the size of the elements, is small or tends to zero, and thereby they are asymptotic studies. The estimator is thus based on appropriate Taylor expansions, and gives in this manner some indications on the admissible size  $h$ . Nevertheless, as this size is not necessarily small, we propose a novel approach which, although closely related to the previous ones, does not assume any particular hypothesis on this parameter, and therefore is probably more justified. Our approach is besides rather similar, in its spirit,

to certain solutions used in a different domain, namely the mesh generation of parametric patches [6]–[8].

Section 2 gives the mathematical formulation of the problem and reviews the related works. Section 3 introduces a new class of measures to quantify the interpolation error depending on the local deformation or curvature of the Cartesian surface corresponding to the solution. A numerical example is illustrated in Section 4 and finally, the last section provides a brief conclusion.

## II. DEFINITION OF THE PROBLEM AND STATE OF THE ART

Let  $\Omega$  be a domain of  $R^d$  (with  $d = 1, 2$  or  $3$ ) and let  $\mathcal{T}$  be a simplicial mesh of  $\Omega$  composed of linear simplices  $P^1$  or quadratic simplices  $P^2$ . We suppose that, in order to solve a problem given in terms of PDEs on  $\Omega$ , we have made a finite element computation on  $\Omega$  using  $\mathcal{T}$ , and we have obtained the scalar solution  $u_{\mathcal{T}}$ . Denoting by  $u$  the exact solution, the problem firstly consists in evaluating the gap  $e_{\mathcal{T}} = u - u_{\mathcal{T}}$  between  $u$  and  $u_{\mathcal{T}}$  representing the error involved by the finite element solution, and secondly deducing (in general by bounding this gap) another mesh  $\mathcal{T}'$  such that the estimated gap between  $u$  and the solution  $u_{\mathcal{T}'}$  using mesh  $\mathcal{T}'$  is bounded by a given threshold. Several points must be more precisely explained:

- how to quantify the gap  $e_{\mathcal{T}}$  between  $u$  and  $u_{\mathcal{T}}$ ?
- how to use the latter information for building a new mesh on which the gap between the corresponding finite element solution and the exact solution is bounded by a given threshold?

The solution  $u_{\mathcal{T}}$  obtained by the finite element method is not interpolating (*i.e.* the solution obtained at the nodes of  $\mathcal{T}$  does not coincide with the exact value of  $u$  at these nodes). Moreover, for each element of the mesh, it cannot be guaranteed that the solution  $u_{\mathcal{T}}$  coincide with the exact value of  $u$  at one point (at least) of the element. Then, it seems difficult to explicitly quantify the gap  $e_{\mathcal{T}}$ . However, the direct study of this gap has been dealt in several works [9]. But, in the general case, its quantification remains an open problem. Consequently, other indirect approaches have been proposed to quantify or rather bound this gap. Let us denote by  $\tilde{u}_{\mathcal{T}}$  the function interpolating  $u$  on the mesh  $\mathcal{T}$

(which is a piecewise linear or quadratic function, depending on the degree of the elements of  $\mathcal{T}$ ) and by  $\tilde{e}_{\mathcal{T}}$  the gap  $u - \tilde{u}_{\mathcal{T}}$  between  $u$  and  $\tilde{u}_{\mathcal{T}}$ , called the *interpolation error* on  $u$  along mesh  $\mathcal{T}$ . To be able to quantify the gap  $e_{\mathcal{T}}$ , we suppose the following relation holds (Céa's lemma):

$$\|e_{\mathcal{T}}\| \leq C \|\tilde{e}_{\mathcal{T}}\|$$

where  $\|\cdot\|$  denotes a norm and  $C$  is a constant not depending on  $\mathcal{T}$ . In other words, we suppose that the finite element error is bounded by the interpolation error. The original problem is then simplified by considering the following problem: given an interpolation  $\tilde{u}_{\mathcal{T}}$  of  $u$  along a mesh  $\mathcal{T}$ , how to build another mesh  $\mathcal{T}'$  for which the interpolation error is bounded by a given threshold? As  $\tilde{u}_{\mathcal{T}}$  can be seen as a discrete representation of  $u$ , the problem now reduces to a characterization of meshes for which the interpolation error is bounded by this threshold. This problem has been the subject of several studies (see for instance [5]) and, in most of them, the examination of a "measure" of the interpolation error provides some constraints associated with the mesh elements. In the context of mesh adaptation methods,  $h$ -methods or size adaptation are particularly relevant, and the constraints are specified in terms of element sizes. In the following, some classical measures of this error are recalled, as well as resulting constraints on the mesh elements.

To quantify the interpolation error, two kinds of measures can be considered: continuous or discrete. A classical continuous measure of this error is the square of the  $L^2$  norm of  $\tilde{e}_{\mathcal{T}}$ :

$$\|\tilde{e}_{\mathcal{T}}\|_{L^2}^2 = \int_{\mathcal{T}} \tilde{e}_{\mathcal{T}}^2 d\omega = \sum_{K \in \mathcal{T}} \|\tilde{e}_K\|_{L^2}^2$$

with  $\|\tilde{e}_K\|_{L^2}^2 = \int_K \tilde{e}_K^2 d\omega$ ,

where  $\tilde{e}_K$  is the interpolation error on each element  $K$  of  $\mathcal{T}$ , and  $d\omega$  is an elementary volume of  $R^d$ . In two dimensions, considering linear elements and assuming that the Hessian  $H_u$  of  $u$  restricted to the elements is constant, Nadler [10] gives an analytical expression of the measure of the interpolation error  $\|\tilde{e}_K\|_{L^2}^2$  on  $K$  as a function of the area  $A$  of  $K$  and the quantities  $d_i = \frac{1}{2} a_i^T H_u a_i$  (second directional derivatives along the edges) where  $a_i$  is the vector joining vertices  $i$  and  $i + 1$  of  $K$ :

$$\int_K \tilde{e}_K^2 dx dy = \frac{A}{180} \left( \left( \sum_i d_i \right)^2 + \sum_i d_i^2 \right).$$

Berzins [11] extends this result in three dimensions (for linear elements) and shows (still assuming that the Hessian

$H_u$  of  $u$  is constant in element  $K$ ) that:

$$\int_K e_{\mathcal{T}}^2 dx dy dz = \frac{V}{420} \left( \left( \sum_i d_i \right)^2 + \sum_i d_i^2 - d_1 d_4 - d_2 d_5 - d_3 d_6 \right),$$

where  $V$  is the volume of  $K$  and quantities  $d_i$  are similar to the 2D case. Berzins deduces from this expression a measure of the quality of the elements, and thus characterizes the mesh. However, it is unclear to interpret this information in terms of element size. The extension of these results to the case of an arbitrary Hessian  $H_u$  remains open. An alternative measure, well suited to problem solving by the finite element method, consists in considering Sobolev norms of  $\tilde{e}_K$ , in particular the  $H^1$  norm whose square is defined by:

$$\|\tilde{e}_K\|_{H^1}^2 = \int_K (\tilde{e}_K^2 + \|\nabla \tilde{e}_K\|^2) d\omega,$$

where  $\nabla$  represents the gradient and  $\|\cdot\|$  is the usual Euclidian norm. In two dimensions and considering linear elements, Zlamal [12], as also Babuska and Aziz [2], independently propose an upper bound of  $\|\tilde{e}_K\|_{H^1}^2$  by the seminorm  $|u|_2$  of the Sobolev space  $H^2$  whose square is defined by:

$$|u|_2^2 = \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2}^2 + 2 \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2}^2 + \left\| \frac{\partial^2 u}{\partial y^2} \right\|_{L^2}^2.$$

Indeed, they show that:

$$\|\tilde{e}_K\|_{H^1}^2 \leq \Gamma(\theta) |u|_2,$$

where  $\Gamma(\theta)$  is a function depending on the diameter of  $K$  and its internal angles. An extension in three dimensions of this relation has been proposed by Krizek [13]. Again, it seems difficult to establish a constraint in terms of element size for this norm. Another measure, which is simpler, consists in considering the  $L^2$  norm of the gradient of  $\tilde{e}_K$ . It is given by:

$$\|\nabla \tilde{e}_K\|_{L^2}^2 = \int_K \|\nabla \tilde{e}_K\|^2 d\omega.$$

An explicit expression of this error measure related to linear elements has been proposed by Bank and Smith [14] in two dimensions in the case where the Hessian  $H_u$  is constant in  $K$ . An approximation of this expression is given by:

$$\|\nabla \tilde{e}_K\|_{L^2}^2 \approx \frac{\sum_i \|a_i\|^2 \sum_i d_i^2}{48 A}.$$

They use this measure for relocating the nodes of the mesh in order to minimize the error.

Among the discrete measures, one can mention the  $L^\infty$  norm of the interpolation error, which is defined by:

$$\|\tilde{e}_K\|_{L^\infty} = \max_{x \in K} |\tilde{e}_K(x)|,$$

where point  $x$  sweeps element  $K$ . Similarly, assuming that Hessian  $H_u$  is constant on each element, Manzi *et al.* [15] propose an approximation of the measure  $\|\tilde{e}_K\|_{L^\infty}$  from an expression of error  $e_K$  given by D'Azevedo and Simpson [3] for linear elements in two dimensions:

$$\|\tilde{e}_K\|_{L^\infty} \approx \frac{\prod_i \delta_i}{16 \det(H_u) A^2},$$

where  $\delta_i = a_i^T |H_u| a_i$ ,  $|H_u|$  being the absolute value of the Hessian of  $u$ . Using this approximation, they show that if the size  $h$  of  $K$  along all directions verifies  $h^T |H_u| h \leq 3\varepsilon$  then  $\|\tilde{e}_K\|_{L^\infty} \leq \varepsilon$ . This size constraint proves well-suited to  $h$ -methods and the results obtained by the authors show the simplicity and the efficiency of this method. In the context of surface triangulation by linear elements, Anglada *et al.* [7] propose, in the general case where the Hessian of  $u$  is arbitrary, an upper bound of  $\|\tilde{e}_K\|_{L^\infty}$  given by:

$$\|\tilde{e}_K\|_{L^\infty} \leq \frac{2}{9} \sup_{x \in K} \|\vec{p}\vec{q}^T H_u(x) \vec{p}\vec{q}\|,$$

where point  $x$  sweeps element  $K$ ,  $p$  is the vertex of  $K$  such that the barycentric coordinate of  $x$  in  $K$  with respect to  $p$  is maximal, and  $q$  the intersection point of the straight line ( $px$ ) with the edge of  $K$  opposite to  $p$ . They infer that the interpolation error is bounded by a threshold if element  $K$  lies in regions defined and centered at the vertices of  $K$ . Therefore, these regions can be defined at every points of the domain and then constitute constraints for the element sizes.

According to the above description of different works on the subject (although this list is far from being exhaustive), a discrete measure (linking error bound and mesh element size) seems more appropriate in the scope of error estimation for mesh adaptation. The following section details this issue.

### III. A NOVEL APPROACH BASED ON SURFACE GEOMETRY

In this section, we recall the approach proposed by [16] which considers solution  $u$  as a Cartesian surface, and we give a new error estimation in the case of anisotropic geometric surface meshing. Let  $\Omega$  be the computational domain,  $\mathcal{T}(\Omega)$  a mesh of  $\Omega$ , and  $u(\Omega)$  the physical solution obtained on  $\Omega$  using the mesh  $\mathcal{T}(\Omega)$ . The couple  $(\mathcal{T}(\Omega), u(\Omega))$  defines a Cartesian surface  $\Sigma_u(\mathcal{T})$ . Given  $\Sigma_u(\mathcal{T})$ , the problem of minimizing the interpolation error consists in defining an optimal mesh  $\mathcal{T}_{opt}(\Omega)$  of  $\Omega$  for which surface  $\Sigma_u(\mathcal{T}_{opt})$  would be as smooth as possible. For this purpose, we propose to locally characterize the surface in the neighborhood of a vertex. Two methods are introduced: the first one, based on local deformation, can be applied for an isotropic adaptation while the second one, based on local curvature, is suitable to an anisotropic adaptation.

#### A. Local deformation of a surface

The main idea consists in locally characterizing the deviation (of order 0) of a surface mesh  $\Sigma_u(\mathcal{T})$  in the neighborhood of a vertex with respect to a reference plane, in particular the tangent plane to the surface at this vertex. This deviation can be quantified by considering the Hessian along the normal to the surface (*i.e.* the second fundamental form of the surface).

Let  $P$  be a vertex of the solution surface  $\Sigma_u(\mathcal{T})$ . Locally, in the neighborhood of  $P$ , this surface admits a parametric representation  $\sigma(u, v)$ ,  $(u, v)$  being the parameters, with  $P = \sigma(0, 0)$ . The Taylor expansion at order 2 to  $\sigma$  in the neighborhood of  $P$  gives:

$$\begin{aligned} \sigma(u, v) &= \sigma(0, 0) + \sigma'_u u + \sigma'_v v \\ &+ \frac{1}{2} (\sigma''_{uu} u^2 + 2\sigma''_{uv} uv + \sigma''_{vv} v^2) + o(u^2 + v^2) e, \end{aligned}$$

where  $e = (1, 1, 1)$ . If  $\nu(P)$  denotes the normal to the surface at  $P$ , then the quantity  $\langle \nu(P), (\sigma(u, v) - \sigma(0, 0)) \rangle$  ( $\langle \cdot, \cdot \rangle$  denoting the dot product) representing the gap between point  $\sigma(u, v)$  and the tangent plane at  $P$ , expressed by:

$$\begin{aligned} \frac{1}{2} (\langle \nu(P), \sigma''_{uu} \rangle u^2 + 2 \langle \nu(P), \sigma''_{uv} \rangle uv + \langle \nu(P), \sigma''_{vv} \rangle v^2) \\ + o(u^2 + v^2), \end{aligned}$$

is therefore proportional to the second fundamental form of the surface for  $u^2 + v^2$  small enough.

The local deformation of the surface at  $P$  is defined as the maximum gap between vertices adjacent to  $P$  and the tangent plane to the surface at  $P$ . If  $(P_i)$  denotes these vertices, then the local deformation  $\varepsilon(P)$  of the surface at  $P$  is given by:

$$\varepsilon(P) = \max_i \langle \nu(P), \vec{PP}_i \rangle.$$

Consequently, the optimal mesh of  $\Omega$  for  $\Sigma_u(\mathcal{T})$  is a mesh whose size at each node  $p$  is inversely proportional to  $\varepsilon(P)$  where  $P = (p, u(p))$ . More formally, the optimal size  $h_{opt}(p)$  associated with a node  $p$  reads:

$$h_{opt} = h(p) \frac{\varepsilon}{\varepsilon(P)},$$

where  $\varepsilon$  denotes the imposed deviation threshold and  $h(p)$  the element size in the neighborhood of  $p$  in mesh  $\mathcal{T}(\Omega)$ .

It can be noticed that the local deformation is a very simple characterization of the local deviation of the surface, which does not require the explicit computation of the Hessian of the solution. The only disadvantage of this measure is that the resulting adaptive meshes can only be isotropic. In the same context (local deviation minimization), the notion of curvature provides a more precise and anisotropic analysis of this deviation.

### B. Local curvature of a surface

The analysis of the local geometric curvature of the surface representing the solution can be used to minimize also the deviation (of order 1) between the tangent planes of the interpolating solution and those of the exact solution. Indeed, in the context of isotropic surface mesh generation, we show [8] that the two deviations of order 0 and 1 of the surface are bounded by a given threshold if, at any point of the surface, the size of the surface elements is proportional to the minimal radius of curvature. Let  $P = (p, u(p))$  be a vertex of  $\Sigma_u(\mathcal{T})$ , let  $\rho_1(P)$  and  $\rho_2(P)$  with  $\rho_1(P) \leq \rho_2(P)$  be the two principal radii of curvature at  $P$ , and let  $(\vec{e}_1^\Sigma(P), \vec{e}_2^\Sigma(P))$  be the two unit vectors in the corresponding principal directions. The ideal size for a surface mesh element at  $P$  is [8]:

$$h_{opt}^\Sigma(P) = \gamma \rho_1(P),$$

where  $\gamma$  is a coefficient depending on the imposed deviation threshold. This size is defined in the tangent plane to the surface at  $P$ . In the reference system  $(P, \vec{e}_1^\Sigma(P), \vec{e}_2^\Sigma(P))$  of this plane, the ideal size in a given direction is a vector  $\vec{v}(P) = h_1^\Sigma \vec{e}_1^\Sigma(P) + h_2^\Sigma \vec{e}_2^\Sigma(P)$  whose components  $h_1^\Sigma$  and  $h_2^\Sigma$  satisfy the following relation:

$$\begin{pmatrix} h_1^\Sigma & h_2^\Sigma \end{pmatrix} \frac{\mathcal{I}_2}{\gamma^2 \rho_1^2(P)} \begin{pmatrix} h_1^\Sigma \\ h_2^\Sigma \end{pmatrix} = 1.$$

This expression, where  $\mathcal{I}_2$  denotes the  $2 \times 2$  identity matrix, represents the equation of a circle with center  $P$  and radius  $\gamma \rho_1(P)$  in the tangent plane to the surface at  $P$ . By an orthogonal projection of this circle in the plane of  $\Omega$ , the size constraint at  $p$  is obtained. If  $\vec{v}_1^\Sigma(p)$  and  $\vec{v}_2^\Sigma(p)$  are the respective orthogonal projections of  $\vec{e}_1^\Sigma(P)$  and  $\vec{e}_2^\Sigma(P)$  in the plane of  $\Omega$ , then this size constraint in the reference system  $(p, \vec{i}, \vec{j})$  ( $\vec{i} = (1, 0)$  and  $\vec{j} = (0, 1)$ ) is given by:

$$\begin{pmatrix} h_1 & h_2 \end{pmatrix} \mathcal{P}^T \frac{\mathcal{I}_2}{\gamma^2 \rho_1^2(P)} \mathcal{P} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1,$$

where  $\mathcal{P} = \begin{pmatrix} \vec{v}_1^\Sigma(p) & \vec{v}_2^\Sigma(p) \end{pmatrix}^{-1}$  and  $(h_1, h_2)$  are the coordinates in the reference system  $(p, \vec{i}, \vec{j})$  of the projection of the ideal size vector  $\vec{v}(P)$  in the plane of  $\Omega$ . This relation defines, among others, a metric (generally anisotropic) at  $p$ .

This metric may produce an important number of elements owing to the isotropic feature of surface elements. To minimize this number of elements, and in the context of anisotropic geometric surface meshing, we have established [17] a relation which is similar to the isotropic case and depends on both principal radii of curvature. Now, the ideal size of the surface elements is given by a metric, called geometric, which can be expressed at a vertex  $P$  of  $\Sigma_u(\mathcal{T})$ :

$$\begin{pmatrix} h_1^\Sigma & h_2^\Sigma \end{pmatrix} \begin{pmatrix} \frac{1}{\gamma^2 \rho_1^2(P)} & 0 \\ 0 & \frac{1}{\eta^2 \rho_2^2(P)} \end{pmatrix} \begin{pmatrix} h_1^\Sigma \\ h_2^\Sigma \end{pmatrix} = 1,$$

where  $\gamma = 2 \sqrt{\varepsilon(2 - \varepsilon)}$ ,  $\eta = 2 \sqrt{\varepsilon \frac{\rho_1(P)}{\rho_2(P)} (2 - \varepsilon \frac{\rho_1(P)}{\rho_2(P)})}$ ,

in which  $\varepsilon$  is the prescribed gap in direction  $\vec{e}_1^\Sigma(P)$ . This relation generally represents an ellipse in the tangent plane to the surface at  $P$  which contained a circle in the isotropic case. Again, by projecting this ellipse in the plane of  $\Omega$ , the corresponding metric at  $p$  in this plane is obtained. This measures also provide a means to control the interpolation error in  $H^1$  norm (bounding the error on the solution but also on its derivatives), and thereby seems more adequate compared to an isotropic measure.

In practice, to compute the local curvature, several steps are necessary. First, at each vertex of the surface mesh, the normal (hence the gradient) is determined by a weighted average of unit normals to the adjacent elements. Then, in the local reference system (composed of the tangent plane and the normal) associated with each vertex, a quadric centered at this vertex and approaching at best the adjacent vertices is built. Afterwards, the Hessian is locally approximated by the Hessian to this quadric. Knowing the gradient and the Hessian of the solution at the nodes of  $\mathcal{T}(\Omega)$ , the curvatures and principal directions at each vertex of surface  $\Sigma_u(\mathcal{T})$  are obtained.

### IV. NUMERICAL EXAMPLE

To illustrate the proposed method, we consider an image of  $700 \times 536$  pixels and the field of its grey levels. Figure 1 shows the original color image, a reproduction of The Adoration of the Magi (circa 1500). Its author was the North Italian Renaissance painter Andrea Mantegna, whose early career was shaped by impressions of Florentine works. The image is firstly represented by a regular grid of  $699 \times 535$  quadrilaterals, defining its initial mesh. The analysis of the local geometric curvature of the Cartesian surface representing the field leads to the determination of an anisotropic geometric size map associated with the initial mesh, in order to bound the interpolation error (here  $\varepsilon = 0.1$ ). Figure 2 (general view) and 3 (close-up) show the adapted anisotropic mesh. This mesh contains 227,557 vertices and 453,265 triangles. It has been realized using the anisotropic adaptive mesh generator BL2D [18]. The resulting interpolation error is 0.085 in average.

### V. CONCLUSION

A novel approach connecting the problem of *a posteriori* error estimation and some techniques of surface meshing has been introduced. It constitutes an alternative method to classical approaches using the Hessian of the solution.



Figure 1. Original color image (painting by Andrea Mantegna, circa 1500).

To illustrate our methodology, a numerical example has been presented. The proposed *a posteriori* error estimation can be used in any computational problem where a static field must be calculated. In the case of dynamic fields, the adaptive computation is constituted by a calculation loop: at each iteration, beginning at the same global initial time and ending at a different time, a combination of the current metric and the previous metrics is applied.

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Figure 2. Adapted mesh corresponding to the interpolation error  $\epsilon = 0.1$ .

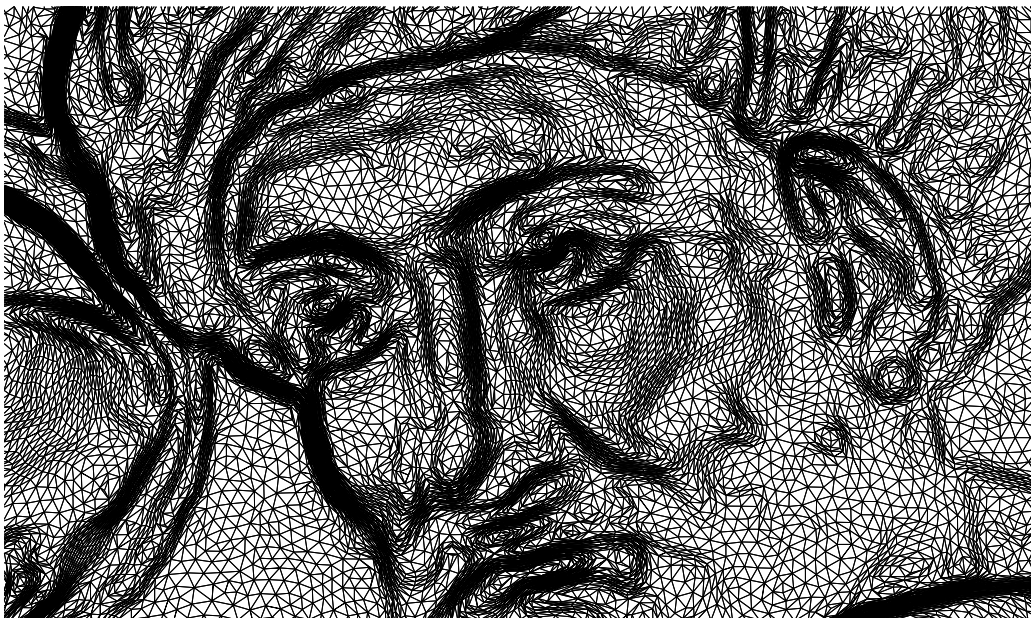


Figure 3. Enlargement of a selected region.