Applications of Random Finite Element Method in Bearing Capacity Problems

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Abstract **– This paper aims to apply a new methodology in bearing capacity analysis, often referred to as random finite element method (RFEM). This method considers the variability of soil parameters within the finite element method (FEM) by generating a Gaussian random field for the parameters within finite elements. The local average subdivision method (LAS) was used in this study to generate the Gaussian random field. However, soil parameters are not, generally, randomly distributed within neighboring soil elements; they tend to be correlated over a distance. Thus, the correlation length, the distance over the soil parameters are correlated to each other, was considered in this study. The Monte Carlo simulation was done for bearing capacity problem and some statistical and probabilistic methods were applied for analyzing the results to get the failure probability of footing on clay. This study would help to understand the effect of variability of soil parameter by using RFEM; so that the safety issues of geotechnical design can be determined in terms of probability of failure.**

Keywords – Random finite element method; Monte Carlo simulations; Gaussian random field generation; Statistical distribution; Probabilistic method; Correlation lengths; Risk assessment.

I. INTRODUCTION

In geotechnical engineering analysis, the soil parameters are often considered as constant within a soil layer. For instance, the bearing capacity of a strip footing is determined using constant value of c and ϕ for each layer, where c is the cohesion and ϕ is the angle of internal friction of soil particle. Thus, the equation for determining the bearing capacity on the surface of clay can be expressed as in Eq. (1) by Terzaghi [1]

$$q_u = cN_c \,. \tag{1}$$

where q_u is bearing capacity, c is cohesion of soil, N_c is bearing capacity factor = 5.14 for ϕ = 0. Eq. (1) gives bearing capacity of clay based on cohesion, c only. Thus, a design based on Eq. (1) can be conservative or optimistic with an associated risk based on how geotechnical engineer determine c for a clay layer. In practice, a value of c is approximated in such a way so that it gives a conservative estimation for design.

However, in reality, c is not the same, i.e., it varies point to point in a soil layer because of complicated geological formation process. The deterministic approach as in Eq. (1) may be conservative, but do not consider realistic condition. Moreover, the implication of this simplification is not explored in details yet. Thus, it is necessary to consider the variation of c in bearing capacity analysis and compare with deterministic approach. So, a new method, (RFEM) Random Finite Element Method was used in this study to calculate bearing capacity.

In RFEM, the material parameters are varied and distributed within finite elements. For example, the parameter c, in the above problem, is varied within finite elements for a clay layer. However, the variation of the parameters should be consistent with field condition. According to Fenton and Griffiths [2], the geotechnical parameters can possibly have several reasonable distributions, which include log-normal, normal and tanh bounded distribution. The parameter c is generally assumed to be log-normally distributed with an advantage of avoiding negative c that has no physical meaning [3, 4]. Then, based on the log-normal distribution, the Gaussian random field of c is generated by Local Average Subdivision, LAS method for finite elements. However, the variation of c within soil elements is not purely random; a smoother change of c between two neighboring soil elements is expected than two elements at a distance apart.

A spatial correlation length is used within the random field to describe the distance over which random values tend to be correlated. When the correlation lengths in horizontal and vertical directions are same, the soil elements can be assumed as isotropic. Most of the previous studies focused on isotropic condition.

This paper focuses on the variability and the effect of anisotropic distribution of material parameters in geotechnical analysis. Some statistical and probabilistic methods, such as random field generator with log-normal distribution, correlation length and Monte Carlo simulations [2] are used within the finite element analysis. The probability of failure of footing on clay was obtained from cumulative distribution function [2].

II. RANDOM FINITE ELEMENT METHOD

RFEM considers that the engineering parameters are distributed over a correlation length as a Gaussian random field [2] generated by local average subdivision [2] within

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finite elements. These techniques are discussed in following subsections.

A. *Correlation length*

In reality, soil parameters within the neighbouring points are similar, i.e., correlated. The distance over that parameter is correlated is called correlation length or scale of fluctuation. According to Fenton and Griffiths [2], the correlation coefficient in isotropy can be determined as

$$\rho(|\tau|) = -2|\tau|/\theta\,. \tag{2}$$

where ρ is correlation coefficient, τ is distance between two points. When the correlation lengths in horizontal and vertical direction are same, then it is called isotropic condition. Most of the previous studies are based on isotropic condition. However, in reality the correlation length in horizontal direction is higher than the vertical direction as geologically soil forms in horizontal layers [5, 6]. The condition, when horizontal and vertical correlation lengths are different, is called anisotropic condition. The anisotropy condition can be expressed as

$$\rho(|\tau_x|,|\tau_y|) = \exp\left\{-\sqrt{(2\tau_x/\theta_x)^2 + (2\tau_y/\theta_y)^2}\right\}. \tag{3}$$

where τ_x and τ_y are the distance in horizontal and vertical direction respectively, θ_x and θ_y are horizontal and vertical correlation lengths, respectively.

The correlation coefficient, ρ, takes an important role in the random field generation. They will take part in LAS process in order to correlate the parameters in finite element meshes. The functions of correlation coefficient are illustrated in Fig.1.

In isotropy, the coefficient is the same in both directions, thus it generates symmetric curve to the centre in both horizontal (τ_x) and vertical directions (τ_y) as shown in Fig. 1a. But, in anisotropy, the correlation coefficient is different in horizontal and vertical directions. In Fig. 1b, when the horizontal correlation length increases to a high value of 100, the correlation length becomes very close to 1.0 in horizontal direction. On the other hand, the vertical correlation coefficient is high at the middle and lower at the side. Hence, the parameters in horizontal direction will correlate better than the vertical direction.

B. *Local average subdivision*

The LAS technique is one of the techniques that widely used to generate the Gaussian random field. It was introduced in Vanmarcke [7]. At first, the global average is generated with mean of zero and unit variance of 1 [2]. The global average is defined by local average theory as in Fenton and Griffiths [2]. The global average can be written in terms of expectation function as in Eq. (4)

$$E[X_T(t)] = E\left[\frac{1}{T}\int_{t-T/2}^{t+T/2} X(\xi)\,d\xi\right]. \tag{4}$$

where T is the mesh size, t is location at the centre of each mesh cell, and ξ is location at moving average. The covariance of the local averages is defined by using a variance function as defined in Fenton and Griffiths [2].

$$\gamma(T) = \frac{2}{T^2}\int_0^T (|T| - \tau)\rho(\tau)\,d\tau\,. \tag{5}$$

where γ = variance function. The variance function indicates the average correlation coefficient, ρ between each pair of 2 separated points within the defined area, where $\rho(\tau)$ is defined in Eq. (3). Then, based on the covariance between pair of cells, the LAS process can be generated. In LAS process, one parent cell, Q is subdivided into 4 equal cells, which are called child cells. The Fig. 2 illustrates the subdivision process, where Q is the parent cell and G is the child cell.

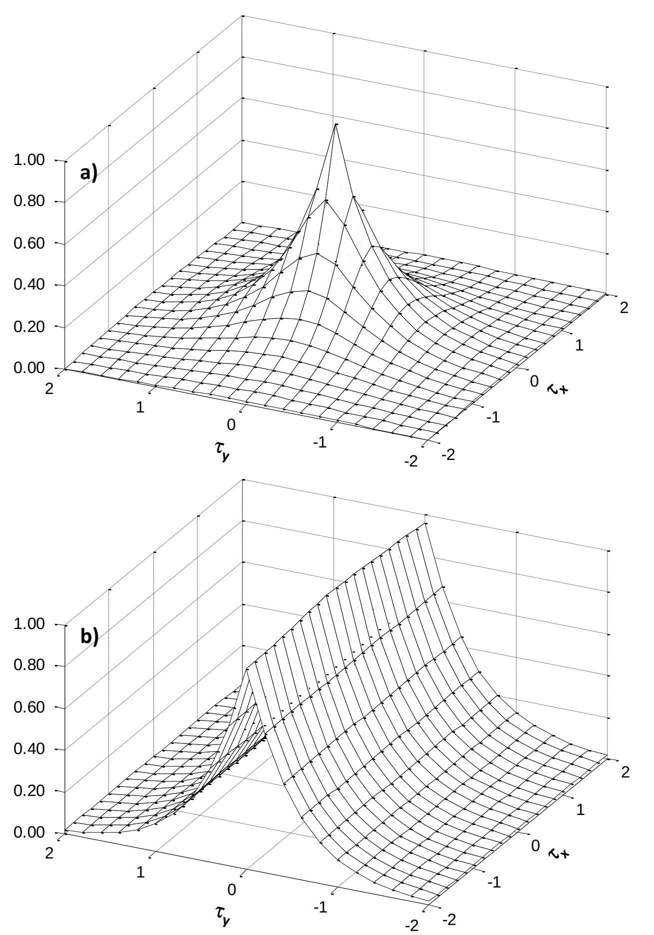

Figure 1. (a) Correlation coefficient in isotropy, $\theta_x = \theta_y = 1.0$ and (b) Correlation coefficient in anisotropy, $\theta_x = 100$ and $\theta_y = 1.0$

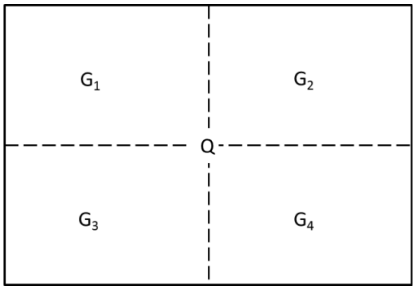

Figure 2. LAS process.

The function which is used for the LAS process can be expressed as

$$G = A^T Q + LU\ . \tag{6}$$

In this process, U is indicated as the vector of independent standard normal random variables with mean zero and unit standard deviation.

The covariance described the relationship between the cells and can be written as the following equations.

Covariance between parent cells

$$R = E\left[QQ^T\right]. \tag{7}$$

Covariance between parent and child cells

$$S = E\left[QG^T\right]. \tag{8}$$

Covariance between child cells

$$B = E\left[GG^T\right]. \tag{9}$$

Then, the matrices A and L can be determined by

$$A = R^{-1}S\ . \tag{10}$$

$$LL^T = B - S^T A. \tag{11}$$

The method used to generate matrix L is called matrix decomposition method, which is used to find the lower triangular matrix from the defined matrix. According to Fenton and Griffiths [2], the LAS technique is most reliable and efficient technique to generate the random field for RFEM and also give the best-fit results to the theory.

C. *Random field transformation*

The cohesion of soil, c, is considered as log-normal distribution and thus the log-normal distribution transformation in Gaussian random field can be expressed as

$$X(i, j) = \exp[\mu + \sigma G(i, j)]\ . \tag{12}$$

where $X(i, j)$ is transformed random field, μ is mean, σ is standard deviation, and $G(i, j)$ is random field generated by LAS process in Eq. (6). A distribution of c is shown in Fig. 3.

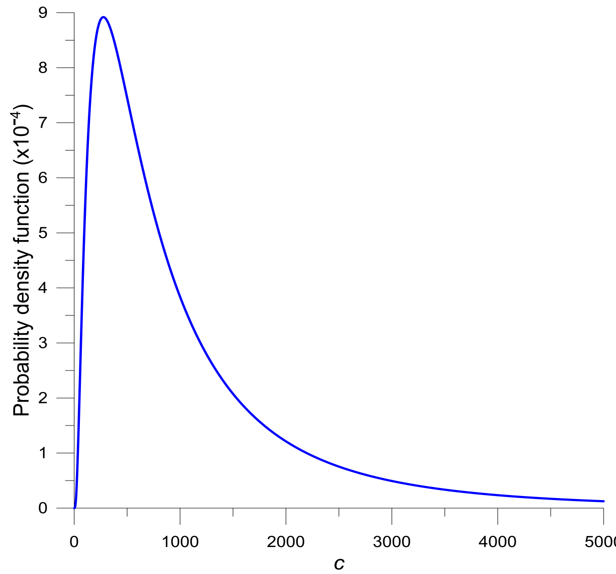

Figure 3. Log-normal distribution with μ=400 and σ=300

D. *RFEM and variability of soil parameters*

An elastic-perfectly plastic stress-strain law with Tresca failure criterion is used in finite element formulation. The theoretical method is described in details in Chapter 6 of the text by Smith and Griffiths [8]. The software used in this study is called Mrbear2D and freely available online. The soil parameters including dilation angle, α, elastic modulus, E and Poisson's ratio, ν are assumed to be deterministic with specific constant values. However, the undrained shear strength parameter, c was a variable in terms of coefficient of variation (COV) can be expressed as

$$COV = \sigma / \mu\ . \tag{13}$$

where μ is mean and σ is standard deviation of c. The distribution of c within finite element meshes for small correlation of $\theta_x = \theta_y = 0.1$ and $\theta_x = 10$ & $\theta_y = 4.0$ are shown in Fig. 4a & b. A smoother variation is apparent for higher correlation length.

III. MONTE CARLO SIMULATIONS

The LAS technique will generate a random field in each Monte Carlo simulation [2]. This type of simulation is applied in order to consider the possible variability of

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parameter in geotechnical analysis. The Monte Carlo simulation process continues with LAS technique and distribution function until simulations obtain stable results. By using RFEM software, a reasonable number of 1000 FEM simulations were done to obtain stable result as shown in Fig. 5.

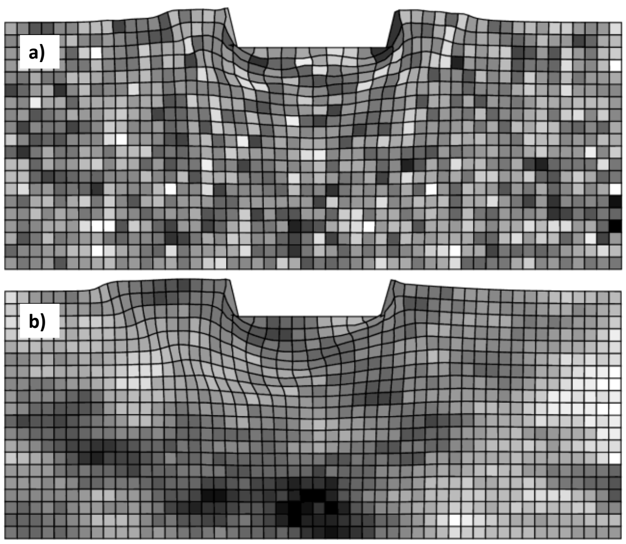

Figure 4. The mesh model with correlation coefficient; (a) $\theta_x = \theta_y = 0.1$ and (b) $\theta_x = 10$ and $\theta_y = 4$

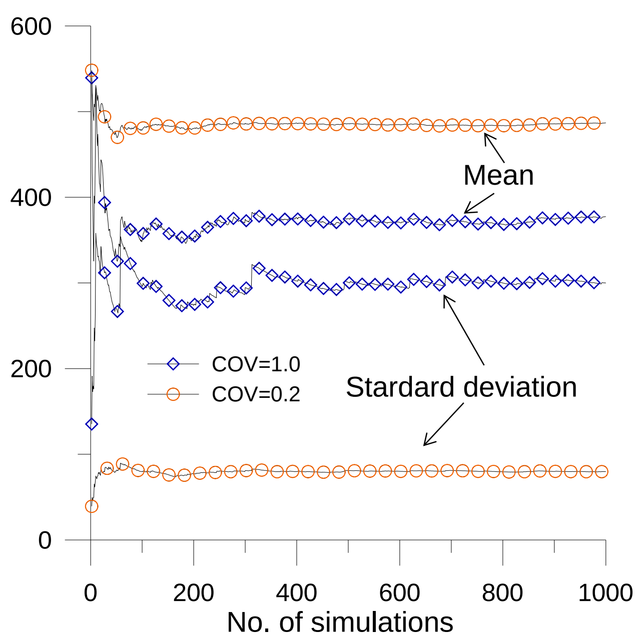

Figure 5. Mean value and standard deviation of the results through Monte Carlo simulations.

The y-axis and x-axis in Fig. 5 present bearing capacity and number of simulations, respectively. A number of 1000 FEM simulations, using Monte Carlo simulation, is adequate for this study. However, the effect of COV on bearing capacity is also apparent in the figure.

IV. RESULTS

When doing the analysis with 1000 Monte Carlo simulations, there are 1000 bearing capacities for 1000 different c fields. In RFEM, the cohesion, c was the input with log-normal distribution. Then, the bearing capacity factor, N_c will be determined by using cohesion and bearing capacity in Eq. (1).

$$N_c = q_u / \mu_c \, . \tag{14}$$

Because c log-normally distributed, N_c then can be considered as log-normally distributed. Thus, the mean and standard deviation of N_c can be expressed as

$$\mu N_c = \sum_{i=1}^{1000} N_{ci} / 1000 \, . \tag{15}$$

$$\sigma N_c = \sqrt{\sum_{i=1}^{1000} (N_{ci} - \mu N_c) / 1000} \, . \tag{16}$$

where N_{ci} is N_c from each simulation, μN_c is mean of 1000 N_{ci}, σN_c is standard deviation of 1000 N_{ci}. As N_c log-normally distributed, the logarithm values of mean and standard deviation of N_c were used in the probabilistic method for calculating the probability of failure. The conventional deterministic method in geotechnical engineering adopted Prandtl solution for the bearing capacity factor, N_c of 5.14 [9]. Thus, the probability of failure is considered as the chance of the mean bearing capacity factor, μN_c less than 5.14 [3, 10]. The probability can be expressed in terms cumulative function, Φ

$$P[N_c < 5.14 / FS] = \Phi(\beta) \, . \tag{17}$$

where β is reliability index and FS is the factor of safety. The factor of safety is applied to minimise the probability of failure of the footing. The reliability index, β of N_c, which is the expression of margin of safety, M from its critical value (M=0) [5] can be defined as

$$\beta = \mu M / \sigma M \, . \tag{18}$$

where μM is mean of margin of safety, σM is standard deviation of margin of safety.

In this case, the margin of safety, M is the difference between the Prandtl solution from deterministic approach and the mean of bearing capacity factor [5]. Thus, mean and standard deviation of M can be expressed as

$$\mu M = 5.14 / FS - \mu N_c . \tag{19}$$

$$\sigma M = \sigma N_c . \tag{20}$$

The probability of failure, hence, can be determined by the cumulative function of β. The cumulative function can be illustrated as Fig.6.

The correlation length in RFEM has impacts on the probability of failure for geotechnical problems [2] and this study considered different correlation lengths both for isotropic and anisotropic cases. The isotropic study with RFEM was mentioned by many authors includes Griffiths and Fenton [3], Vessia *et al.* [6] and Popescu *et al.* [11]. The probability of failure for isotropic condition for *FS*=3.0 and *COV* =1.0 is compared with other published data in Fig. 7. A good match is observed with Griffiths and Fenton [3], however significant discrepancies are observed with Kasama and Whittle [10]. It is worth nothing that this study used displacement based finite element method whereas Kasama and Whittle [10] used numerical limit analysis. However, the differences are not obvious and need further investigation. The probability of failure for anisotropic cases is also plotted in Fig. 7, θ_x = 1, 2 and 4 are plotted with varying θ_y/B (*B* is width of footing). It shows that the probability of failure for anisotropic cases can be higher than the isotropic case irrespective of the method used in Griffiths and Fenton [3] and Kasama and Whittle [10]. However, the probability of failure in any case is not higher than 25%.

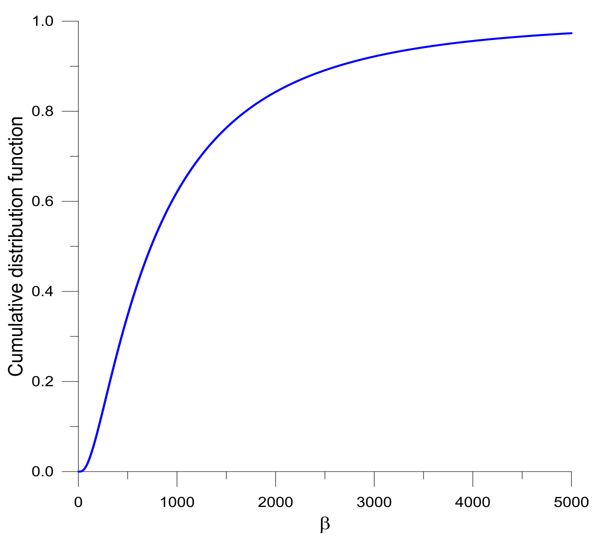

Figure 6. Cumulative density function of β

For better understanding the probability of failure in anisotropic condition, a contour map is presented for *COV*=0.1 in Fig. 8, where red is higher and blue is lower probability of failure. In Fig. 8, the probability of failure is very high at the small correlation length and the probability of failure decreases when correlation length increases. Generally, the probability of failure is affected by the correlation length and *COV* of *c* in RFEM. When the correlation length is high, the probability is low. In contrast, the *COV* is high; the probability of failure will be high. In anisotropy, the ratio of correlation length is increasing while the probability of failure is decreasing. It is interesting to note that the higher correlation length in horizontal direction (θ_x) with θ_y in the range of 1 to 3 is better in terms of stability when comparing with higher correlation length in vertical direction (θ_y). This is favorable as this is more realistic for general field condition. However, the opposite condition is not usual in field conditions though it is not impossible.

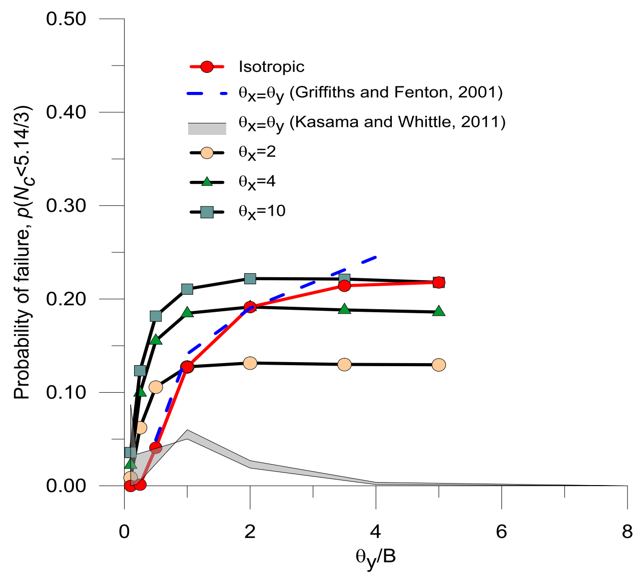

Figure 7. Probability of failure against vertical correlation length (*COV*=1.0)

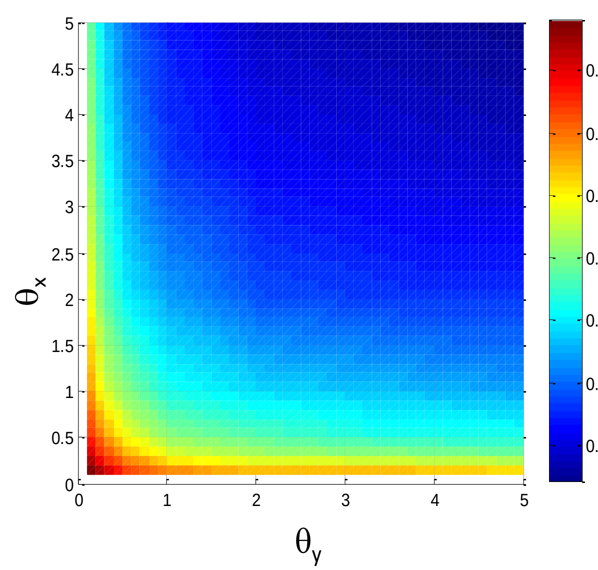

Figure 8. Probability of failure contour plot for COV=0.1 against horizontal and vertical correlation length.

As mentioned previously, this study considered higher horizontal correlation length than the vertical direction and this is presented as the ratio of the horizontal and vertical correlation length, θ_x/θ_y. The effect of θ_x/θ_y on the probability of failure is shown in Fig. 9.

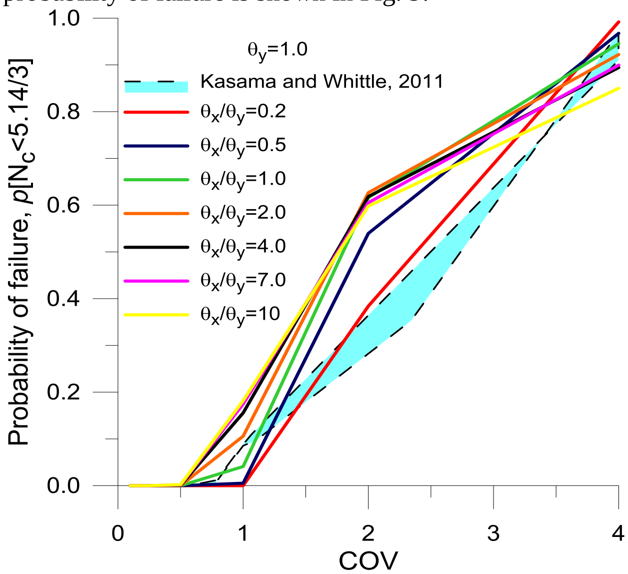

Figure 9. Probability of failure against COV

Fig. 9 shows that the probability of failure will be increasing when the *COV* of *c* increases. It is obvious that the results from anisotropy are significantly different than the isotropic results in Kasama and Whittle [10]. The results in both conditions do not match well when *COV*=2.0 to 3.0. For anisotropic conditions, at small *COV*, the greater the ratio of correlation length is, the higher the probability of failure is. In contrast, at high *COV*, the greater the ratio is, the smaller the probability of failure is.

V. CONCLUSION

This paper discussed about the random finite element method (RFEM) and applied in bearing capacity problem. Then, some probabilistic and statistical methods used to evaluate the effect of the variability of soil parameter on the probability of footing failure. The correlation length of soil parameter within neighbouring soil elements and its effect on the probability of failure is explored. The major findings of this study are-

- The probability of failure for isotropic condition is different for different methods. This is not obvious, thus need further investigation.
- The probability of failure for anisotropic conditions is higher than the isotropic conditions for the cases presented in the study.
- A higher correlation length in horizontal direction is more favourable than a higher correlation length vertical direction. This is favourable to most of the general conditions.
- The factor of safety is also considered in the calculation of the probability of failure. The suitable factor of safety for footing design is 3.0, when the probability of failure is significantly small, particularly at the small *COV* and correlation length.

RFEM considers the variation of soil parameters within soil elements. It is more practical and realistic than deterministic approaches, which considers parameters are constant for all soil elements. By using RFEM, the chance of failure of footing can be determined more realistically. The probability of failure will vary against different correlation length and coefficient of variation (*COV*). In this case, RFEM can improve the accuracy and efficiency in geotechnical analysis when dealing with a distribution of parameters.

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