The Zero-Sum Tensor

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***Abstract*—The zero-sum matrix, or in general, tensor, reveals some consistent properties at multiplication. In this paper, three mathematical rules are derived for multiplication involving such entities. The application of these rules may provide for a more concise and straightforward way to formulate mathematical proofs that rely on such matrices.**

Keywords-matrix; multiplication; n-simplex; tensor; zero-sum

I. Introduction

On the topic of rare matrices, some properties of a matrix, here defined as a *zero-sum matrix*, are analyzed and three rules are derived governing multiplication involving such matrices. The suggested category (the zero-sum matrix) does not seem to presently exist, and is as expected neither included in lists, such as [3]. In this paper, a *zero-sum matrix* is defined as a matrix where the sum of the column vectors is equal to a zero column vector and/or the sum of the row vectors is equal to a zero row vector, or in the general case, a zero-sum tensor of size $N_1 \times N_2 \times \cdots \times N_Q$, where summation along one, or several dimensions, results in a P-dimensional tensor (with $P = Q - 1$), that consists of zero-elements only. This rule applies to any tensor T of dimension $Q \in \mathbb{N}_2$ (all integers equal or greater than two). A matrix where the sum of the columns and rows both are equal to zero vectors, could further be defined as a *complete* zero-sum matrix, and similarly in the general case, a *complete* zero-sum tensor could be defined as a tensor where summation along all dimensions results in a P-dimensional tensor that consists only of zero-elements.

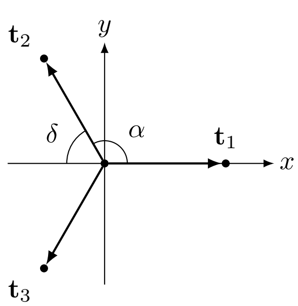

Figure 1. An example with a 2-simplex matrix $\mathbf{T}_2 = [\mathbf{t}_1\ \mathbf{t}_2\ \mathbf{t}_3]^T$, with the dihedral angle $\delta = \pi - \alpha$.

An example of a zero-sum matrix is a regular n-simplex matrix, based on the n-dimensional geometric object called the n-simplex. A few examples are the 0-simplex (point), the 1-simplex (line segment), the 2-simplex (triangle) and the 3-simplex (tetrahedron). If the object is fully symmetric (all edges are of equal length), it is called *regular*. Scaled appropriately, the regular n-simplex exhibits the following properties:

$$\mathbf{t}_i \cdot \mathbf{t}_j = \begin{cases} 1, & i = j \\ -1/n, & i \neq j \end{cases} \tag{1}$$

$$\sum_{i=1}^{n+1} \mathbf{t}_i = \mathbf{0} \tag{2}$$

where $\mathbf{t}_i$ and $\mathbf{t}_j$ with $i, j \in \{1, 2, \ldots, N\}$ and $N = n + 1$ denote any unit vectors i and j pointing from the center of the regular n-simplex to its i:th and j:th vertices. These properties were confirmed in [4] and [6] in context with an elementary mathematical proof of the relation $\delta = \arccos(\frac{1}{n})$, where δ denotes the dihedral angle of the regular n-simplex. For $n = 1$, $t_1 = -t_2 = 1$. For $n = 2$, as shown in Fig. 1:

$$\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{t}_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \quad \mathbf{t}_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \tag{3}$$

In this example, the vectors $\mathbf{t}_i$ spanning the coordinate system of the regular n-simplex, are placed so that $\mathbf{t}_1$ coincides with the x-axis.

As a brief overview, we start by the derivation of three mathematical rules, followed by an example for the demonstration of Rule III (which is slightly more complex than the other two), and finally conclude, by the application of Rule III to reconfirm an already existing mathematical proof.

II. General Case

The idea behind the derivation of the rules presented in this paper originated from the evaluation of $\mathbf{H} = \mathbf{T}^T\mathbf{T}$ in [1], with the proposition of the extension of minimax [5], and alpha-beta pruning [2], from the two-person case to the general N-person case, which as a side effect led to the discovery of a new elementary method for the calculation of the dihedral angle of the regular n-simplex. The relation in (2), was used in this context by Fridenfalk [1], to derive a generic algorithm for the recursive calculation of $\mathbf{T} = [\,\mathbf{t}_1\ \mathbf{t}_2 \ldots \mathbf{t}_N\,]$, for $N \in \mathbb{N}_2$, with $N = n + 1$:

$$\left. \begin{aligned} t_{ii} &= \sqrt{1 - \sum_{j=1}^{i-1} \gamma_j^2} \\ \gamma_i &= -\frac{t_{ii}}{n + 1 - i} \end{aligned} \right\} 1 \leq i \leq n \tag{4}$$

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$$\mathbf{T} = \begin{bmatrix} 1 & \gamma_1 & \gamma_1 & \cdots & \gamma_1 & \gamma_1 & \gamma_1 & \gamma_1 \\ 0 & t_{22} & \gamma_2 & \cdots & \gamma_2 & \gamma_2 & \gamma_2 & \gamma_2 \\ 0 & 0 & t_{33} & \cdots & \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t_{(n-2)(n-2)} & \gamma_{n-2} & \gamma_{n-2} & \gamma_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & t_{(n-1)(n-1)} & \gamma_{n-1} & \gamma_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & t_{nn} & \gamma_n \end{bmatrix} \tag{5}$$

An alternative and concise proof of the relation in (2) follows by the derivation of Rule III in this paper. Before the presentation of this rule, we start by the establishment of two basic rules.

Rule I. Given the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{H}$ of size $M \times N$, $N \times K$, and $M \times K$, respectively, such that $\mathbf{H} = \mathbf{AB}$, if sum $\mathbf{a}$ of the row vectors of $\mathbf{A}$, is equal to a zero row vector $\mathbf{0}_N^T$ of size $1 \times N$, then sum $\mathbf{u}$ of the row vectors of $\mathbf{H}$, is equal to a zero row vector $\mathbf{0}_K^T$ of size $1 \times K$.

Proof. Given:

$$\mathbf{u} = \begin{bmatrix} b_{11}(a_{11}+\ldots+a_{M1})+\ldots+b_{N1}(a_{1N}+\ldots+a_{MN}) \\ b_{12}(a_{11}+\ldots+a_{M1})+\ldots+b_{N2}(a_{1N}+\ldots+a_{MN}) \\ \vdots \\ b_{1K}(a_{11}+\ldots+a_{M1})+\ldots+b_{NK}(a_{1N}+\ldots+a_{MN}) \end{bmatrix}^T \tag{6}$$

$\mathbf{a} = \mathbf{0}_N^T \rightarrow \mathbf{u} = \mathbf{0}_K^T$.

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Rule II. Given the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{H}$ of size $M \times N$, $N \times K$, and $M \times K$, respectively, such that $\mathbf{H} = \mathbf{AB}$, if sum $\mathbf{b}$ of the column vectors of $\mathbf{B}$ is equal to a zero column vector $\mathbf{0}_N$ of size $N \times 1$, then sum $\mathbf{v}$ of the column vectors of $\mathbf{H}$ is equal to $\mathbf{0}_M$ of size $M \times 1$.

Proof 1. Given:

$$\mathbf{v} = \begin{bmatrix} a_{11}(b_{11}+\ldots+b_{1K})+\ldots+a_{1N}(b_{N1}+\ldots+b_{NK}) \\ a_{21}(b_{11}+\ldots+b_{1K})+\ldots+a_{2N}(b_{N1}+\ldots+b_{NK}) \\ \vdots \\ a_{M1}(b_{11}+\ldots+b_{1K})+\ldots+a_{MN}(b_{N1}+\ldots+b_{NK}) \end{bmatrix} \tag{7}$$

$\mathbf{b} = \mathbf{0}_N \rightarrow \mathbf{v} = \mathbf{0}_M$.

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Proof 2. Given Rule I and the rules for matrix transpose, $\mathbf{H} = \mathbf{AB} \Leftrightarrow \mathbf{H}^T = \mathbf{B}^T\mathbf{A}^T$, thus, $\mathbf{b} = \mathbf{0}_N \rightarrow \mathbf{v} = \mathbf{0}_M$.

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Rule III. Given the real matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{H}$ of size $M \times N$, $N \times M$, and $M \times M$, respectively, such that $\mathbf{H} = \mathbf{AB}$, if $\mathbf{A} = \mathbf{B}^T$, $\mathbf{v}$, defined as the sum of the column vectors of the symmetric matrix $\mathbf{H}$, is equal to a zero column vector $\mathbf{0}_M$ of size $M \times 1$, then $\mathbf{b}$, defined as the sum of the column vectors of $\mathbf{B}$, is equal to a zero column vector $\mathbf{0}_M$ of size $M \times 1$.

Proof. Given (7), $\mathbf{1}_M = [1 \quad 1 \quad \ldots \quad 1]^T$ of size $M \times 1$ and s, a positive-definite scalar, equal to the sum of the elements of the symmetric matrix $\mathbf{H}$:

$$s = \mathbf{1}_M^T\mathbf{H} \cdot \mathbf{1}_M = \mathbf{1}_M^T\mathbf{v} \tag{8}$$

as $a_{kj} = b_{jk}$ and $\mathbf{v} = \mathbf{0}_M \rightarrow s = \sum_{j=1}^{N}(b_{j1} + b_{j2} + \ldots + b_{jM})^2 = 0 \rightarrow \mathbf{b} = \mathbf{0}_N$, since:

$$s = \mathbf{1}_M^T\mathbf{v} = 0 \Rightarrow \begin{cases} b_{11} + b_{12} + \ \ldots \ + b_{1M} = 0 \\ b_{21} + b_{22} + \ \ldots \ + b_{2M} = 0 \\ \vdots \\ b_{N1} + b_{N2} + \ldots + b_{NM} = 0 \end{cases} \tag{9}$$

Thus, $\mathbf{v} = \mathbf{0}_M \rightarrow \mathbf{b} = \mathbf{0}_N$.

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Once $\mathbf{u}$ and $\mathbf{v}$ are derived, as shown in (6)-(7), the derivation of the first two rules is straightforward. To concretize, the following example demonstrates the third rule for a 2×3 matrix, $\mathbf{B} = \mathbf{A}^T$. Given:

$$\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \tag{10}$$

If $\mathbf{H} = \mathbf{AB}$, $\mathbf{1}_3 = [1 \quad 1 \quad 1]^T$ and $s = \mathbf{1}_3^T\mathbf{H} \cdot \mathbf{1}_3 = \mathbf{1}_3^T\mathbf{v}$, then:

$$\begin{aligned} s &= \mathbf{1}_3^T \begin{bmatrix} a(a+b+c) + d(d+e+f) \\ b(a+b+c) + e(d+e+f) \\ c(a+b+c) + f(d+e+f) \end{bmatrix} \\ &= (a+b+c)^2 + (d+e+f)^2 \end{aligned} \tag{11}$$

Thus, $\mathbf{v} = \mathbf{0}_3 \rightarrow s = (a+b+c)^2 + (d+e+f)^2 = 0 \rightarrow \mathbf{b} = \mathbf{0}_2$, since $s = 0 \rightarrow a+b+c = d+e+f = 0$. Or in other words, if $\mathbf{H} = \mathbf{B}^T\mathbf{B}$ is a 3×3 zero-sum matrix, then the sum of the columns of $\mathbf{B}$ is a zero-vector of size 2×1.

III. Application

As an example of the application of Rule III, given (1), a new and a more straightforward proof of (2) is hereby produced:

Proposition. The sum of the unit vectors $\mathbf{t}_i$ of a regular n-simplex, where each vector i points from the center of the object to its i:th vertex, is equal to $\mathbf{0}$.

Proof. Given a n-simplex matrix $\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2 \ldots \mathbf{t}_{n+1}]$, $\mathbf{H} = \mathbf{T}^T\mathbf{T}$ (of size $n+1 \times n+1$), and:

$$h_{ij} = \mathbf{t}_i \cdot \mathbf{t}_j = \begin{cases} 1 & i = j \\ -1/n & i \neq j \end{cases} \tag{12}$$

given (1), where h_{ij} denotes an element in $\mathbf{H}$. Thus, the sum of any row (or column) in $\mathbf{H}$ is equal to $1 - \frac{1}{n} \cdot n = 0$, and since $\mathbf{H}$ is a (symmetric) zero-sum matrix, according to Rule III, $\sum_{i=1}^{n+1} \mathbf{t}_i = \mathbf{0}$.

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IV. CONCLUSION

In this paper, a *zero-sum matrix* (or tensor) has been closely defined, along with the related concept *complete*. Three rules have been presented governing multiplications involving such entities, along with an example of the application of Rule III for a concise reconfirmation of (2), exemplified in this paper by a proposition.

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