Balanced Reduction of an IDE-based Spatio-Temporal Model

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***Abstract*—Spatio-temporal models have the potential to represent a wide variety of dynamic behaviour such as the growth of bacteria, the dispersion of a pollutant or the changing spatial patterns in house prices. Classical methods for the simulation of such behaviours suffer from large computational demands due to their high dimensionality. Recent advances in spatio-temporal modelling have proposed a method based on a state-space representation of the spatio-temporal integro-difference equation. Although the dimension reduction obtained when using this model is significant, it is frequently not sufficient for online computation or rapid simulation. Thus this model is revisited in this work and a method for further dimension reduction based on a balanced realization of the state-space model is developed. The results will show that the computational cost reduction obtained is significant at the expense of a minor loss in accuracy.**

***Keywords*-spatio-temporal simulation, balanced model reduction, integro-difference equation.**

I. Introduction

Various natural phenomena in a wide spectrum of scientific disciplines exhibit complex interactions over both space and time. These interactions are particularly common in biology, ecology, meteorology, epidemiology, physics, environmental science and economics. Broad ranging methods have been used to describe spatio-temporal behaviour. For example, in physics, reaction-diffusion processes have been successfully modelled via Coupled Map Lattices (CML) with parameters estimated directly from data [1], [2]. Geostatistical spatio-temporal models have also been estimated from data in both ecological applications such as the monitoring of pollution [3] and in meteorology for, among others, the modelling for rainfall [4] and wind behaviour [5], [6]. In epidemiology, Auto-Regressive Moving-Average (ARMA) models have been used to describe the diffusion of fowl-pest diseases [7] while hierarchical Bayesian models have been used to analyze geographic disease rates [8].

A problem common to most spatio-temporal models is their usually high dimensionality leading to large computational demands for simulation and prediction. Certain methods suffer from further limitations such as the need of CML to have data measured on a regular grid; an impractical condition in applications such as meteorology and epidemiology. Moreover, CMLs also require some knowledge of the natural laws involved to propose an adequate model structure. This knowledge is not always at hand when modelling complex behaviour common for ecological or meteorological applications. Finally, even when measurements are taken on a regular grid, it is often required to infer estimates at other locations among the measurement sites. Unfortunately, most modelling strategies do not provide efficient and rigorous methods to perform such spatial interpolations.

A promising mathematical description of spatio-temporal behaviour that has the potential to overcome or minimize the effect of these limitations is the Integro-Difference Equation (IDE) [9], [10]. In this representation, the spatio-temporal dynamics are governed by a convolution integral in space and a difference equation in time, with the spatio-temporal dynamics dictated by a convolution kernel. In some recently proposed representations, the evolving field modelled by the IDE is decomposed into a set of weighted basis functions also used to decompose the convolution kernel [11], [12]. These decompositions allow the approximate representation of the IDE by a finite dimension state-space model. This framework has the advantage of decoupling the number of states from observation locations with the potential of overcoming the dimensionality issues hampering various other models. Moreover, since in the proposed methods the convolution kernel is completely estimated from data, no prior knowledge of the natural laws involved is required. Finally, since the spatio-temporal behaviour is represented by a basis function decomposition, spatial interpolation is both computationally efficient and mathematically sound.

Other recent additions to this IDE-based spatio-temporal model have proposed a method based on spectral analysis to identifying an adequate number of basis functions to represent some measured behaviour [13], [14]. Nevertheless, the models obtained may still suffer from large computational demands when the spatial bandwidth and/or the spatial domain under investigation are large. Thus in this work a method from systems theory is used to further reduce the dimensionality of the model obtained. The errors introduced by this order reduction procedure will be given analytically and shown experimentally.

The remainder of this paper is organised as follows.

 ISBN: 978-1-61208-159-5

In Section II the state-space representation of the IDE is presented, followed by the dimension reduction procedure and the errors introduced in Section III. Section IV expands on the advantages of the proposed method based on a synthetic example. Finally, Section V gives some concluding remarks and possible future enhancements.

II. STATE-SPACE REPRESENTATION OF THE IDE

Consider the spatially continuous, temporally discrete spatio-temporal process $z(\mathbf{s},t) \in \mathbb{R} : \mathbf{s} \in \mathcal{S} \subset \mathbb{R}^n, t \in \mathbb{Z}^+$ where $n = \{1,2,3,\ldots\}$, $\mathcal{S}$ is a fixed spatial domain and $\mathbf{s}$ and t are spatial and temporal indexes, respectively.

Definition 1. The temporally Markovian, spatially homogeneous, time invariant, Gaussian spatio-temporal IDE is given by

$$z(\mathbf{s},t) = \int_{\mathcal{S}} k(\mathbf{s}-\mathbf{r}) z(\mathbf{r},t-1)\,\mathrm{d}\,\mathbf{r} + \eta(\mathbf{s},t) \tag{1}$$

where $k(\mathbf{s}-\mathbf{r}) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a spatially homogeneous convolution kernel and $\eta(\mathbf{s},t)$ is a zero mean stationary Gaussian noise process with covariance Σ_η given by

$$\Sigma_\eta = \mathrm{COV}[z(\mathbf{s},t), z(\mathbf{s}+\mathfrak{s}, t+\mathfrak{t})] = \begin{cases} \lambda(\mathfrak{s}) & \text{if } \mathfrak{t} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Remarks.

1) *The spatio-temporal dynamics of the system are governed by the shape of the convolution kernel. The choice on the space of the kernel is dictated by the process under investigation; for example, simple reaction-diffusion processes can be modelled by Gaussian kernels [9].*
2) *Although a spatially homogenous, time-invariant kernel will be considered here, heterogenous, time-varying kernels can easily be incorporated in the representation as shown in [11].*
3) *The temporal dynamics are here limited to first order Markovian, this assumption can be lifted by including higher order terms with different convolution kernels.*

The stochastic process $z(\mathbf{s},t)$ is observed via a number of identical noisy sensors located at $\{\mathbf{s}_i,\ i = 1,2,\ldots,n_y\}$ to obtain the data-set $Y = \{y_t,\ t = 1,2,\ldots,T\}$ where $y_t = [y(\mathbf{s}_1,t)\ y(\mathbf{s}_2,t)\ \ldots\ y(\mathbf{s}_{n_y},t)]^\top$. Each sensor can be characterized by the spatial convolution

$$y(\mathbf{s}_i,t) = \int_{\mathcal{S}} h(\mathbf{s}_i-\mathbf{r}) z(\mathbf{r},t) d\,\mathbf{r} + v(t) \tag{3}$$

where $h(\mathbf{s}_i-\mathbf{r})$ is the spatial response of the sensors used and $v(t)$ is a zero mean white Gaussian noise process uncorrelated with $\eta(\mathbf{s},t)$.

The direct computational representation of the stochastic process $z(\mathbf{s},t)$ is intractable due to the continuous nature of the spatial domain. To overcome this problem, [11], [12] have suggested a method based on basis function decompositions of the stochastic process, the convolution kernel, the spatial response of the sensor and the noise covariance to obtain an approximate discrete state-space representation of the IDE. In these methods, the state-space dimension is given by the number of basis functions used to decompose the dynamic field. In [13] a method based on spectral analysis and multi-dimensional extensions of Shannon sampling theorem have been used to obtain an initial estimate of the number, position and parameters of the basis functions used for the decomposition. Joint estimation of the stochastic process and the convolution kernel from noisy data can then be performed by a variety of methods such as the dual Kalman filter [15], the Expectation Maximization (EM) algorithm [16] or in a Bayesian setting, by a 2-stage Gibbs sampler [17].

Such a state-space representation of the IDE requires that the Assumptions 1 and 2 are satisfied.

Assumption 1 (Spatial Low-Pass Response)**.** The spatio-temporal process $z(\mathbf{s},t)$ must exhibit a spectral low-pass behaviours, that is:

$$Z(\boldsymbol{\nu},t) \approx 0\ \forall\ t,\ \ \boldsymbol{\nu} \notin \mathcal{V} \tag{4}$$

where $Z(\boldsymbol{\nu},t)$ is the Fourier transforms of $z(\mathbf{s},t)$ and $\mathcal{V} = [0,\nu_c]^n$, with ν_c being the spatial cut-off frequency.

Assumption 2 (Spatial Semi-Compact Support)**.** The spatio-temporal process $z(\mathbf{s},t)$ must be semi-compactly supported, that is:

$$z(\mathbf{s},t) \approx 0\ \forall\ t,\ \ \mathbf{s} \notin \mathcal{S}\,. \tag{5}$$

Remarks.

1) *Assumption 1 implies that the spatio-temporal process must exhibit some spatial smoothness. Such a condition is generally satisfied by most practical processes.*
2) *Assumption 2 implies that the spatial domain under observation must be finite, again a condition that is usually satisfied in most spatio-temporal studies.*

Using the basis function approximations

$$z(\mathbf{s},t) \approx \sum_{j=1}^{n_x} \langle z(\mathbf{s},t), \phi_{x_j}(\mathbf{s})\rangle \phi_{x_j}(\mathbf{s}) = \mathbf{x}(t)^\top\, \boldsymbol{\phi}_x(\mathbf{s}) \tag{6}$$

$$k(\mathbf{s}) \approx \sum_{j=1}^{n_\theta} \langle k(\mathbf{s}), \phi_{\theta_j}(\mathbf{s})\rangle \phi_{\theta_j}(\mathbf{s}) = \boldsymbol{\theta}^\top\, \boldsymbol{\phi}_\theta(\mathbf{s}) \tag{7}$$

$$h(\mathbf{s}) \approx \sum_{j=1}^{n_\vartheta} \langle h(\mathbf{s}), \phi_{\vartheta_j}(\mathbf{s})\rangle \phi_{\vartheta_j}(\mathbf{s}) = \boldsymbol{\vartheta}^\top\, \boldsymbol{\phi}_\vartheta(\mathbf{s}) \tag{8}$$

$$\lambda(\mathbf{s}) \approx \sum_{j=1}^{n_\varrho} \langle \lambda(\mathbf{s}), \phi_{\varrho_j}(\mathbf{s})\rangle \phi_{\varrho_j}(\mathbf{s}) = \boldsymbol{\varrho}^\top\, \boldsymbol{\phi}_\varrho(\mathbf{s}) \tag{9}$$

 ISBN: 978-1-61208-159-5

where

$$\begin{aligned}\mathbf{x}(t) &= [\langle z(\mathbf{s},t),\phi_{x_1}(\mathbf{s})\rangle \quad \dots \quad \langle x(\mathbf{s},t),\phi_{x_{n_x}}(\mathbf{s})\rangle]^\top \\ \boldsymbol{\theta} &= [\langle k(\mathbf{s}),\phi_{\theta_1}(\mathbf{s})\rangle \quad \dots \quad \langle k(\mathbf{s}),\phi_{\theta_{n_\theta}}(\mathbf{s})\rangle]^\top \\ \boldsymbol{\vartheta} &= [\langle h(\mathbf{s}),\phi_{\vartheta_1}(\mathbf{s})\rangle \quad \dots \quad \langle h(\mathbf{s}),\phi_{\vartheta_{n_\vartheta}}(\mathbf{s})\rangle]^\top \\ \boldsymbol{\varrho} &= [\langle \lambda,(\mathbf{s})\phi_{\varrho_1}(\mathbf{s})\rangle \quad \dots \quad \langle \lambda(\mathbf{s}),\phi_{\varrho_{n_\varrho}}(\mathbf{s})\rangle]^\top \\ \boldsymbol{\phi}_i(\mathbf{s}) &= [\phi_{i_1}(\mathbf{s}) \ \dots \ \phi_{i_{n_i}}(\mathbf{s})]^\top\end{aligned} \tag{10}$$

and $\phi_i(\mathbf{s})$ are some chosen basis functions, an approximate state-space representation of the IDE with known error bounds is given by Theorem 1.

Theorem 1. *Using the spatially discrete representations (6) to (9) with Assumptions 1 and 2 satisfied, the stochastic IDE of Definition 1 and the observation equation (3) can be approximated by the finite dimension state-space model*

$$\mathbf{x}(t+1) = A(\boldsymbol{\theta})\,\mathbf{x}(t) + \mathbf{w}(t) \tag{11}$$

and

$$\mathbf{y}(t) = C(\boldsymbol{\vartheta})\,\mathbf{x}(t) + \mathbf{v}(t) \tag{12}$$

where

$$A(\boldsymbol{\theta}) = \Psi^{-1} \int_{\mathcal{S}} \boldsymbol{\phi}_x(\mathbf{s})\,\boldsymbol{\theta}^\top\,\Xi_\theta(\mathbf{s})d\,\mathbf{s} \tag{13}$$

$$\Psi = \int_{\mathcal{S}} \boldsymbol{\phi}_x(\mathbf{s})\,\boldsymbol{\phi}_x(\mathbf{s})^\top d\,\mathbf{s} \tag{14}$$

$$\Xi_\theta(\mathbf{s}) = \int_{\mathcal{S}} \boldsymbol{\phi}_\theta(\mathbf{s}-\mathbf{r})\,\boldsymbol{\phi}_x(\mathbf{r})^\top d\,\mathbf{r} \tag{15}$$

$$C(\boldsymbol{\vartheta}) = \begin{pmatrix} \boldsymbol{\vartheta}^\top\,\Xi_\vartheta(\mathbf{s}_1) \\ \vdots \\ \boldsymbol{\vartheta}^\top\,\Xi_\vartheta(\mathbf{s}_{n_y}) \end{pmatrix}$$

$$\Xi_\vartheta(\mathbf{s}) = \int_{\mathcal{S}} \boldsymbol{\phi}_\vartheta(\mathbf{s}-\mathbf{r})\,\boldsymbol{\phi}_x(\mathbf{r})^\top d\,\mathbf{r} \tag{16}$$

$$\mathbf{w}(t) \sim \mathcal{N}(0,\Sigma_w) \tag{17}$$

with

$$\Sigma_w = \Psi^{-1} \int_{\mathcal{S}} \boldsymbol{\phi}_\varrho(\mathbf{s})\varrho(\mathbf{s})^\top \Xi_\varrho(\mathbf{s})d\,\mathbf{s}\,\Psi^{-\top} \tag{18}$$

$$\Xi_\varrho(\mathbf{s}) = \int_{\mathcal{S}} \boldsymbol{\phi}_\varrho(\mathbf{s}-\mathbf{r})\,\boldsymbol{\phi}_x(\mathbf{r})^\top d\,\mathbf{r} \tag{19}$$

and $\mathbf{v}(t) \sim \mathcal{N}(0,\Sigma_v)$ *with* $\Sigma_v = \sigma_v I_{n_y}$*; with errors in the approximation of* $z(\mathbf{s},t)$ *given by*

$$\epsilon_z = |z(\mathbf{s},t) - \mathbf{x}(t)^\top\,\boldsymbol{\phi}_x(\mathbf{s})| \leqslant \epsilon'_z \int_{\mathbb{R}^n:\boldsymbol{\nu}>\boldsymbol{\nu}_c} \Phi_x(\boldsymbol{\nu})d\,\boldsymbol{\nu} \tag{20}$$

where

$$\epsilon'_z = \sup_{\mathbb{R}^n:\boldsymbol{\nu}>\boldsymbol{\nu}_c} |Z(\boldsymbol{\nu})\Phi_x^{-1}(\boldsymbol{\nu})| \tag{21}$$

Remarks.

1) Proof of Theorem 1 is given in [13].

2) The given error bounds assume that a closed form solution for (14), (15), (16) and (19) exist. Such a condition is satisfied by Gaussian basis functions.

3) The state evolution equation (11) of the state-space model of Theorem 1 can be rewritten as

$$\mathbf{x}(t+1) = A(\boldsymbol{\theta})\,\mathbf{x}(t) + B\dot{\mathbf{w}}(t) \tag{22}$$

where $\dot{\mathbf{w}}(t)$ *is a zero-mean Gaussian white noise process with covariance* $\Sigma_{\dot{w}} = I_{n_x}$ *and* B *being the Cholesky decomposition of* Σ_w*, that is* $BB^* = \Sigma_w$ *where* B^* *denotes the conjugate transpose of* B*.*

The computation cost of simulating a spatio-temporal process using this model depends on the state-space model dimension. Based on Theorem 1, good approximations can only be obtained if the full spatial extent and the full bandwidth are considered. This often results in computationally expensive models. A choice to limit the system bandwidth can be taken, but this results in the spatial smoothing of the field estimates and predictions. As an alternative, Section III presents a dimension reduction method based on a balanced realization of the state-space model.

III. Dimension Reduction

Balanced reduction methods of state-space models rely on two steps:

1) The original state-space model is first transformed into a balanced realisation with the states most effected by noise being also the most observable states.
2) The states that are less effected by noise in the balanced realisation (and therefore also less observable) are removed to obtain an approximate truncated state-space model.

The initial transformation into a balanced realization requires that all states of the state-space model are both perturbable and observable. This allows for a linear transformation of the state-space model into a balanced realization. This balanced model can then be truncated to retain only the perturbable and observable states. The perturbability and observability requirements are ensured if conditions 1 and 2 are satisfied.

Condition 1 (Perturbability)**.** All the states of the state-space model of Theorem 1 are perturbable iff, the matrix

$$P = [B \ AB \ A^2B \ \dots \ A^{n_x-1}B] \tag{23}$$

is of full rank.

Condition 2 (Observability)**.** All the states of the state-space model of Theorem 1 are observable iff, the matrix

$$O = [C \ CA \ CA^2 \ \dots \ CA^{n_x-1}]^\top \tag{24}$$

is of full rank.

Remarks.

 ISBN: 978-1-61208-159-5

1) For the state-space model of the IDE of Theorem 1, conditions 1 and 2 are easily satisfied by a well spread arrangement of both the sensors and basis functions used to represent the dynamic field. Such an arrangement requires that no two identical sensors or basis functions are positioned at the same spatial location.

Given that these conditions are satisfied, a balance realization of the state-space model of the IDE is given by Lemma 1.

Lemma 1 (Balance Realization)**.** *If conditions 1 and 2 are satisfied, then a balance realization of the states space model of Theorem 1 is given by:*

$$\breve{\mathbf{x}}(t+1) = \breve{A}\breve{\mathbf{x}}(t) + \breve{B}\hat{\mathbf{w}}(t) \tag{25}$$

and the observation equation

$$\mathbf{y}(t) = \breve{C}\breve{\mathbf{x}}(t) + \mathbf{v}(t) \tag{26}$$

where $\breve{\mathbf{x}}(t) = T\,\mathbf{x}(t)$, $\breve{A} = TAT^{-1}$, $\breve{B} = TB$, $\breve{C} = CT^{-1}$ *and* $T \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ *is a linear transformation such that the matrices* P *and* O *of the transformed model satisfy* $P^\top P = O^\top O = \Upsilon$*, where*

$$\Upsilon = \mathrm{diag}(\sigma_1,\ \sigma_2\ \ldots\ \sigma_{n_y}) \tag{27}$$

and where $\{\sigma_i, i = 1, 2, \ldots\}$ *are the Hankel singular values of the state-space model with* $\sigma_1 > \sigma_2 > \sigma_3 \ldots$.

Remarks.

1) Proof of Lemma 1 is given for general state-space models in [19].
2) Conditions 1 and 2 ensure that the linear transformation T *exists.*
3) Standard matrix computation packages provide accurate methods for obtaining the transformation matrix T*. These methods are mostly based on the contragradient algorithm [18].*

A reduced order model with known error bounds based on the balanced realization of Lemma 1 is given in Theorem 2

Theorem 2. *Using the balanced state-space model of Lemma 1, an approximate reduced order state-space model is given by:*

$$\tilde{\mathbf{x}}(t+1) = \tilde{A}\tilde{\mathbf{x}}(t) + \tilde{B}\tilde{\mathbf{w}}(t) \tag{28}$$

and

$$\mathbf{y}(t) = \tilde{C}\tilde{\mathbf{x}}(t) + \mathbf{v}(t) \tag{29}$$

where $\breve{\mathbf{x}}(t) = [\tilde{\mathbf{x}}(t)^\top \ldots]^\top$, $\hat{\mathbf{w}}(t) = [\tilde{\mathbf{w}}(t)^\top \ldots]^\top$, *with* $\tilde{\mathbf{x}}(t), \tilde{\mathbf{w}}(t) \in \mathbb{R}^{n_r}$ *and*

$$\breve{A} = \begin{pmatrix} \tilde{A} & \cdots \\ \vdots & \ddots \end{pmatrix} \tag{30}$$

$$\breve{B} = \begin{pmatrix} \tilde{B} & \cdots \\ \vdots & \ddots \end{pmatrix} \tag{31}$$

$$\breve{C} = \begin{pmatrix} \tilde{C} & \ldots \end{pmatrix} \tag{32}$$

where $\tilde{A} \in \mathbb{R}^{n_r} \times \mathbb{R}^{n_r}$, $\tilde{B} \in \mathbb{R}^{n_r} \times \mathbb{R}^{n_r}$ *and* $\tilde{C} \in \mathbb{R}^{n_y} \times \mathbb{R}^{n_r}$*, with a maximum error between the impulse responses of the two systems denoted by* ϵ *and given by*

$$\epsilon = 2(\sigma_{n+1} + \sigma_{n+2} + \ldots) \tag{33}$$

Remarks.

1) Proof of Theorem 2 is given for general state-space models in [19].
2) As $n_r \to n_x$*, the maximum error bound is reduced at the cost of a higher dimensional model and thus higher computational demands.*

IV. EXAMPLE

To illustrate the advantages and assess the error introduced by the proposed model reduction procedure, a synthetic data-set was generated by the IDE of Definition 1 and the observation process (3) using numerical integration. All functions and parameters of the IDE and the observation equation are as given in Table I.

Function or parameter		Simulation Value
$\mathcal{S}$	$\in$	$[-6, 6]$
t	$\in$	$[0, 10]$
$k(s)$	=	$0.35\exp(-s^2) + 0.2\exp(-(s-1)^2)$
$\lambda(s)$	=	$\delta(s)$
$z(s,0)$	=	$\frac{1}{\sqrt{4\pi}}\exp(\frac{-(s)^2}{4})$
$h(s)$	=	$\frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{1}{2}\frac{\mathbf{s}^2}{0.7})$
σ_v^2	=	0.01
n_y	=	25 (equally spaced)

Table I
IDE AND OBSERVATION EQUATION FUNCTIONS AND PARAMETERS.

Given the functions and parameters of Table I, a process realisation generated by the IDE is shown in Figures 1.

A first approximate state-space representation of the IDE as given in Theorem 1 is obtained. The state-space and decomposition parameters chosen are as given in Table II.

Function or Parameter		Simulation Value
n_x	=	13
$\phi_x(s)$	=	$\exp\left(-\frac{s}{0.4}\right)$
basis locations for $z(s,t)$	=	$\{-6,\ -5,\ \ldots,\ 6\}$

Table II
STATE-SPACE MODEL ORDER, BASIS FUNCTIONS AND BASIS LOCATIONS.

Based on this representation, the stochastic field shown in Figure 1 is approximated by its discrete basis function reconstruction $\hat{z}_1(s,t)$ shown in Figure 2.

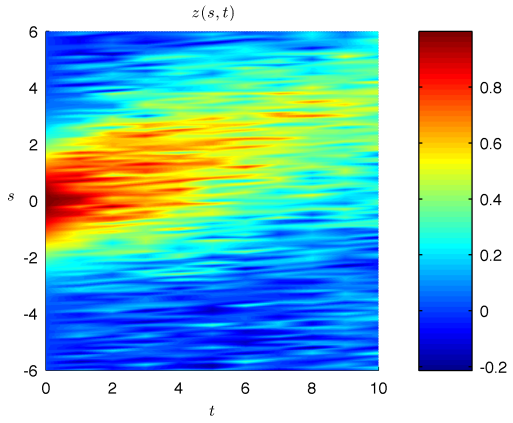

Figure 1. Typical spatio-temporal process $z(\mathbf{s},t)$ generated by first order numerical integration.

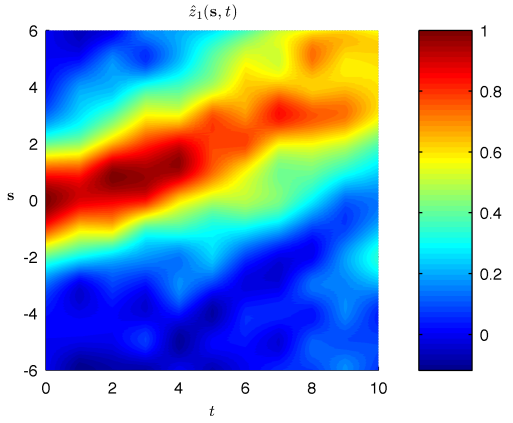

Figure 2. Approximate reconstructed spatio-temporal process $\hat{z}_1(\mathbf{s},t)$ using the discretized model.

Note that the spatially discrete model has a state-space dimension of $n_x = 13$. Thus its system behaviour is captured by a 13×13 matrix which is used in all the computations. Given this state-space representation and the model reduction method of Theorem 2, a reduced order model was obtained with all balanced states with Hankel singular values $\sigma_i < 0.1$ removed. This reduced order model has a state-space dimension of $n_x = 6$ and therefore obtains a significant reduction in computational costs since operations are now performed on 6×6 matrices. This reduced order model generates the approximate field $\hat{z}_2(s,t)$ shown in Figure 3.

The error field $e(s,t)$ showing the error in $\hat{z}_2(s,t)$ introduced by the model reduction procedure when compared to the discretized model field $\hat{z}_1(s,t)$ is shown in Figure 4. This error field indicates that the model reduction procedure has eliminated some higher frequency components but still produced a reasonable approximation to the original stochastic field.

The Root Mean Squared Error (RMSE) between $\hat{z}_1(s,t)$ and $\hat{z}_2(s,t)$ for the example being considered is 0.051. To verify the repeatability of this result, a Monte Carlo run of 100 different stochastic realizations was performed obtaining the RMSE spread shown in Figure 5, with a mean and standard deviation given by 0.0498 ± 0.0048.

Figure 3. Approximate reconstructed spatio-temporal process $\hat{z}_2(\mathbf{s},t)$ using the reduced order model.

Figure 4. Error field $e(s,t)$ between $\hat{z}_1(s,t)$ and $\hat{z}_2(s,t)$.

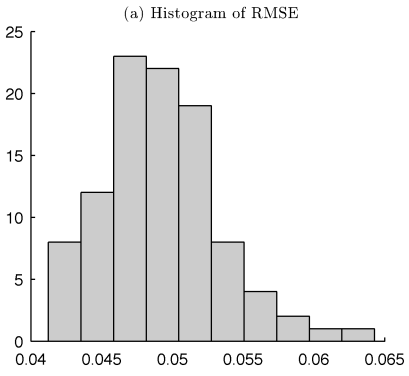

Figure 5. Histogram of RMSE values.

The histogram of Figure 5, indicates the excellent re-

peatability of the results obtained. Moreover, an error of approximately 0.05 is equivalent to 8% of the average measured observation; a remarkable accuracy considering the 54% reduction in the state-space dimension.

V. Conclusion

Mathematical models of spatio-temporal phenomena are continuously gaining in popularity in various scientific fields. Such models are fundamental for mathematical simulation and therefore prediction of spatio-temporal behaviour. However, these models are frequently severely hampered by the high computation demands of most spatio-temporal simulation methods. Thus in this paper a simulation method based on balanced model reduction of an IDE-based spatio-temporal model is given. The simulation results show the ability of the proposed method to represent spatio-temporal behaviour accurately with significant reductions in the computational costs.

Further work is currently being carried out on this computational method to enhance its applicability to varied applications. While a non-linear growth term, as required in biomedical and ecological applications, has already been included [14], heterogeneous and time-varying implementations have still to be developed. Moreover, applying this computational method to various engineering applications, such as fluid dynamics and mechanical structure analysis, requires the inclusion of boundary conditions. Such additions are also being investigated.

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 ISBN: 978-1-61208-159-5