Population Inversion by an Adiabatic Cycle in 1D Boson System

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***Abstract*—In this paper, we show that an adiabatic cycle excites Bose particles confined in a one-dimensional box. We consider a cycle, during which a wall described by a δ-shaped potential is applied and its strength and position are slowly varied. When the system is initially prepared in the equilibrium ground state, the adiabatic cycle brings all bosons into the first excited one-particle state, leaving the system in a nonequilibrium state. The absorbed energy during the cycle is proportional to the number of bosons.**

Keywords–Quantum mechanics; Quantum holonomy; Ultracold gases.

I. INTRODUCTION

The control of a physical system in at quantum level is at the forefront of today's physics research. Among the research works, the population inversion in bosonic gas has a venerable tradition, dating back to the early years of quantum mechanics. The quantum population inversion has been studied for its application to lasing [1]. The quantum control of atoms and molecules has also been investigated to realize the population inversion [2][3]. Recently, studies of the super-Tonks-Girardeau gas, which also involves the population inversion, have attracted a lot of attention in both experimental and theoretical studies of nonequilibrium cold atoms [4][5][6][7]. In the super-Tonks-Girardeau gas, which may be described by the Lieb-Liniger model [8] with strongly attractive interaction, the population inversion is created through an "adiabatic" process, where the interaction strength is suddenly flipped from infinitely repulsive to infinitely attractive [6][9].

Such a population inversion can be induced even by an adiabatic cycle, which can be obtained with an extension of the adiabatic process that connects Tonks-Girardeau and super-Tonks-Girardeau gases both to weaker repulsive and weaker attractive regime. The repetitions of this adiabatic cycle transform the ground state of non-interacting bosons into their higher excited states and achieve the population inversion [10]. This is counterintuitive, since there is no external field to drive the final state of the bosons away from the initial state.

There have been studies of the excitation of quantum systems by adiabatic cycles, which is referred to as the exotic quantum holonomy [11][12][13]. We also mention, in studies of atomic and molecular systems under the oscillating field, that an adiabatic cycle involving a level crossing may excite a quantum system [14][15].

In this paper, we examine an adiabatic cycle that excites a system consisting of Bose particles confined in a one-dimensional box. During the cycle, we vary an additional wall adiabatically, while the interparticle interaction is kept fixed. This is in contrast to the scheme described in [6][10], where the interaction strength between Bose particles is an effective adiabatic parameter. In this study, we suppose that the wall is described by a δ-function shaped potential [16][17][18]. We show that the first excited one-particle state is occupied by all the bosons to achieve the population inversion completely, if the system is prepared to be in the ground state. Namely, the energy gained by the bosons during the adiabatic cycle is proportional to the number of bosons.

The paper is organized as follows. In section 2, the basic setup of our model is presented in the simplest setting of one boson. Section 3 and 4 deal with noninteracting and interacting many bosons, respectively. The summary and the discussion are found in Section 5.

II. A PARTICLE IN A BOX WITH A δ-WALL

In order to examine N Bose particles in a one-dimensional box with an additional δ-wall, we review the single particle case, i.e., $N = 1$ [18], where the system is described by the Hamiltonian

$$H(g, X) = \frac{p^2}{2m} + V(x) + g\delta(x - X), \tag{1}$$

where m is the particle mass, $V(x)$ is the confinement potential, and g and X are the strength and position of δ-wall. In particular, we assume that $V(x)$ describes an infinite square well with the length L, i.e., $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ otherwise [16][17].

We introduce an adiabatic cycle C, which consists of three adiabatic processes C_I, C_{II} and C_{III}, as shown in Figure 1. We suppose that the δ-wall is initially absent, i.e., $g = 0$ in 1, and that the system is in a stationary state initially. In the first part of C, which will be called C_I, an impenetrable wall is inserted at x_0 adiabatically. In terms of the δ-wall, the strength g is slowly increased from 0 to ∞, while its position X is fixed at x_0 during C_I. Subsequently, in the second part C_{II}, the position X of the impenetrable wall is adiabatically changed

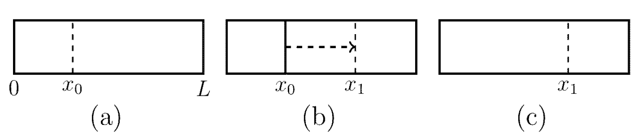

Figure 1. The adiabatic cycle C of a one-dimensional box, which contains Bose particles. The strength and the position of an additional δ-wall is adiabatically varied during C.

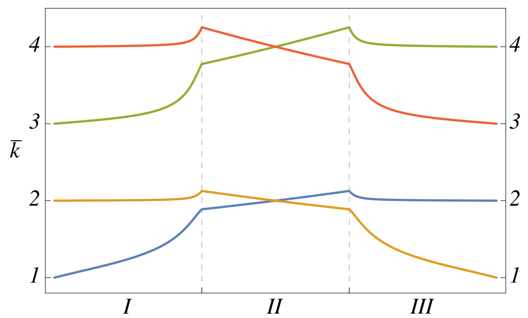

Figure 2. Parametric evolution of eigenenergies with $N = 1$ along the cycle C, which consists of C_I (left part), C_{II} (middle part) and C_{III} (right part). The eigenenergies are depicted by their normalized wavenumber $\bar{k}$ 2. We set $x_0 = 0.4703L$ and $x_1 = L - x_0$.

from x_0 to x_1. In the last part C_{III}, the δ-wall at $X = x_1$ is adiabatically turned off. At the end of the cycle C, the δ-wall has no effect, again.

In Figure 2, we depict the parametric dependence of eigenenergies of the single-particle Hamiltonian $H(g, X)$ 1 along C. Throughout this manuscript, we indicate the eigenenergy E using a normalized wavenumber $\bar{k}$

$$\bar{k} \equiv \sqrt{\frac{E}{N\epsilon}}, \tag{2}$$

where $\epsilon = (\hbar\pi/L)^2/(2m)$ is the ground eigenenergy of the particle in the infinite square well.

The adiabatic time evolution of the single-particle system along C depends on x_0 and x_1. In the following, we explain the case $\frac{2}{5}L < x_0 < \frac{1}{2}L < x_1 < \frac{3}{5}L$, which may be explained from Figure 2. A more rigorous argument is found in [18].

First, let us consider the case that the initial state is the ground state $|1(g = 0, X = x_0)\rangle$ of the particle in the infinite square well, where $|n(g, X)\rangle$ denotes the n-th adiabatic eigenstate of $H(g, X)$ during processes C_I and C_{III}. We will omit to indicate (g, X) in the following. After the completion of C_I, the state vector arrives at $|R_1\rangle$, the ground state of the right well, since we choose the right well in C_{II} is slightly larger than the left well. During C_{II}, there occurs a spectral degeneracy between $|R_1\rangle$ and $|L_1\rangle$, the ground state of the left well. This is because the size of the left (right) well is increasing (decreasing) during C_{II}, and these sizes coincide at $X = L/2$. At the end of C_{II}, $|R_1\rangle$ becomes the first excited state, which adiabatically continued to $|2\rangle$, which is the second excited state of the particle in the infinite square well, through C_{III}. Hence, the "population inversion" in the single-particle system occurs if the system is prepared to be in the ground state initially.

Second, we examine the case that the initial state is the first excited state $|2\rangle$, which offers the "inverse" of the population inversion. Through the adiabatic cycle C, the system arrives at $|L_1\rangle$ after the completion of C_1, and then arrives at $|1\rangle$ at the end of the cycle C. Namely, either $|1\rangle$ and $|2\rangle$ return to the initial states after the completion of the adiabatic cycle C twice.

Third, let us examine the cases that the initial states are $|3\rangle$ and $|4\rangle$, which are the first and second excited state, respectively. Now C induces an interchange of these two states, through the intermediate states $|R_2\rangle$ and $|L_2\rangle$, which are localized the right and left well during the process II.

A similar interchange of initial eigenstates occurs as a result of the adiabatic cycle C, as long as we choose x_0 and x_1 appropriately. In general, the level crossing of the one-particle Hamiltonian 1 during the process C_{II} plays an important role to determine which pairs of eigenstates are interchanged by C, while there is no level crossing generically during the processes C_I and C_{III} [18].

We make a remark on the stability of the present scheme for the one-body population inversion. A crucial point is the stability of the adiabatic time evolution across the level crossing during C_{II}. The level crossing may be lifted due to an imperfection of the impenetrable wall, i.e., the δ-wall with an infinite strength. If the level splitting is small enough, we may employ the diabatic process around the avoided crossing to realize the one-body population inversion. It has been shown that an open diabatic process made of time-dependent potential well produces the second excited state from the ground state of a bose particle [19]. This diabatic scheme is applied to create collective excitations of interacting bosons [19][20][21].

III. Non-interacting Bosons

We examine the case that the number of the Bose particles is N, assuming the absence of interparticle interaction. It is straightforward to extend the above result for $N = 1$, once we restrict the case that N bosons initially occupy the one-particle state $|n\rangle$. Hence, the system is in an adiabatic state of the N bosons

$$|n^{\otimes N}\rangle \equiv |nn\ldots n\rangle, \tag{3}$$

where the one-particle adiabatic state $|n\rangle$ is occupied by N bosons, during C_I and C_{III}.

If there are no interparticle interactions, the parametric evolution of averaged wavenumber $\bar{k}$ for the adiabatic N-particle state agrees with the one of the single-particle system. This suggests that the adiabatic cycle C of the N-particle system with no interaction delivers the ground state $|1^{\otimes N}\rangle$ to the excited state $|2^{\otimes N}\rangle$, i.e., the complete population inversion, as seen in Figure 2. The energy that the particles acquire during the cycle C is proportional to the number of the particles.

IV. Interacting Bosons

We examine the adiabatic cycle C for N interacting Bose particles. We mainly examine the case that the system is initially in the ground state. In order to confirm that the N-particle population inversion really occurs, we need to examine the effect of the interparticle interaction.

We assume that the interparticle interaction V consists of two-body contact interactions. Namely, we suppose that V takes the following form

$$V(x_1, x_2, \ldots, x_N) = \lambda \sum_{\langle i,j \rangle} \delta(x_i - x_j), \tag{4}$$

where λ is the interaction strength, and the summation is taken over pairs.

We also assume that the interparticle interaction is weak enough so that the topology of the parametric dependence of eigenenergy remains unchanged, except around the level crossing points of the noninteracting bosons. Namely, when the gaps of the eigenenergies between neighboring levels in the noninteracting system are larger than a constant value, the interparticle interaction shifts the eigenenergy at most $\mathcal{O}(\lambda)$, according to the standard perturbation theory. For small enough perturbative energy correction, the corresponding adiabatic time evolution of the stationary state of the interacting bosons closely follows the one of the noninteracting bosons.

Accordingly, under the weak interparticle interaction condition, the eigenstates of the interacting bosons can be labeled by the quantum numbers of the noninteracting bosons. For example, the ground state of the initial and final points of the adiabatic cycle C may be denoted as $|1^{\otimes N}(\lambda)\rangle$, whose overlapping integral with the unperturbed state $|1^{\otimes N}\rangle$ is large. Also, $|1^{\otimes N}(\lambda)\rangle$ can be constructed by the standard perturbation theory with a small parameter λ.

On the other hand, even a weak interparticle interaction can strongly influence the parametric evolution of energy levels in the vicinity of level crossings by making avoided crossings. Hence, we need to closely examine the level crossing of the non-interacting Bose particles.

In the following, we argue that the adiabatic time evolution closely follows the one in the noninteracting system examined above, if the number of particles is large enough. The key is the selection rule for the matrix element of V in the adiabatic representation in the vicinity of the level crossings of non-interacting Bosons.

A. *"Tunneling" and direct contributions of the interaction in $N = 2$*

We show that the effect of the interparticle interaction is significantly different, depending on whether a level crossing locates either in C_{II}, or in $C_I \cup C_{III}$, as for the two body case. In the former case, the relevant matrix elements may be small since it involves only tunneling processes through the impenetrable wall. On the other hand, in processes C_I and C_{III}, the matrix element cannot be negligible. However, it turns out that there happens to be no corresponding level crossing that affects the population inversion whose initial state is the ground state.

The parametric evolutions of eigenenergies of the noninteracting two particle system are depicted in Figure 3, in terms of the averaged wavenumber $\bar{k}$ (see, 2). The parametric evolution of the eigenenergy that connects $|11\rangle$ and $|22\rangle$ has a level crossing with two eigenenergies during C_{II}. The initial states of these energy levels are $|22\rangle$ and $|12\rangle$, which are $|L_1 L_1\rangle$ and $|R_1 L_1\rangle$ during C_{II}, respectively.

We examine the matrix elements of the interparticle interaction term V between the adiabatic basis vectors $|R_1 R_1\rangle$, $|L_1 L_1\rangle$ and $|R_1 L_1\rangle$. Note that $|R_1 R_1\rangle$ corresponds to the initial state $|11\rangle$ of the adiabatic cycle

$$\langle R_1, L_1|V|R_1, R_1\rangle = \sqrt{2}\lambda\int_0^L \{\psi_{R_1}(x)\psi_{L_1}(x)\}^* \{\psi_{R_1}(x)\}^2 dx, \tag{5}$$

for example. Since the single-particle adiabatic eigenfunctions $\psi_{L_1}(x)$ and $\psi_{R_1}(x)$ are completely localized in the left and right wells, respectively, the overlapping integral is zero, if the δ-wall is completely impenetrable during C_{II}. The level crossing accordingly remains even in the presence of the interparticle interaction. Thus the adiabatic cycle C induces

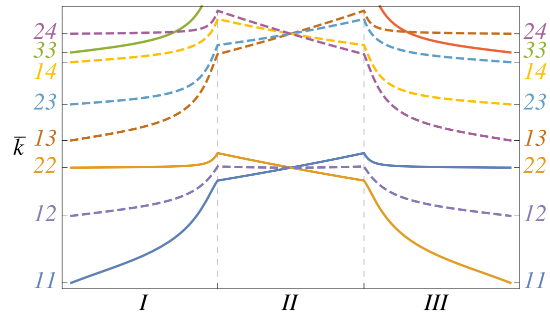

Figure 3. Parametric evolution of normalized wavenumbers for $N = 2$. Full lines indicate the levels whose initial states are $|nn\rangle$ $(n = 1, \ldots, 4)$ Other levels are depicted by dashed lines. Parameters are the same as in Figure 2.

the complete population inversion from $|11\rangle$ to $|22\rangle$, as in the non-interacting case.

Let us examine the case that the δ-wall during C_{II} allows the tunneling leakage of particles due to some imperfections. Still, we may expect that the matrix elements due to the tunneling corrections are exponentially small. Since the resultant energy gap of the avoided crossing is also exponentially small, we may expect that the diabatic process easily almost recovers the complete population inversion.

Also, during the second process C_{II}, the left and right part of the well may be separated. This allows us to make the tunneling correction arbitrarily small. Accordingly the adiabatic limit that follows the extremely small avoided crossing would be difficult to realize.

On the other hand, if the level crossing appears during C_I or C_{III}, the interparticle interaction destroys the level crossing. In Figure 3, such an example is seen between the levels whose initial states are $|33\rangle$ and $|24\rangle$, which is delivered to $|R_2 R_2\rangle$ and $|L_1 L_2\rangle$, respectively, in the absence of V.

The matrix element $\langle 33|V|24\rangle$ does not vanish in general, since the relevant single-particle adiabatic eigenfunctions extend the whole box. Accordingly the level crossings are destroyed to form avoided crossing. Thus the adiabatic process C_I for example, delivers $|33\rangle$ and $|24\rangle$ at the initial point of C_I, to $|L_1 L_2\rangle$ and $|R_2 R_2\rangle$, respectively. This breaks the population inversion whose initial state is a higher excited state, e.g., the adiabatic cycle C delivers $|33\rangle$ to $|44\rangle$ in the absence of the interparticle interaction.

B. *Selection rule for $N = 3$*

Here, we show that the interparticle interaction does not suppress the population inversion for $N > 2$ due to a selection rule of V.

We explain this with the case $N = 3$ (Figure 4). Let us examine the level whose initial state is $|1^{\otimes 3}\rangle$ along C. The corresponding final state is $|2^{\otimes 3}\rangle$ in the absence of the interparticle interaction.

First, the interparticle interaction has no, or exponentially small effect on the level crossing during C_{II}, as shown in the case of $N = 2$.

Second, we examine the level crossing in C_{III}, where the levels whose final states are $|2^{\otimes 3}\rangle$ and $|113\rangle$ exhibit crossing. We examine the matrix element of the interparticle

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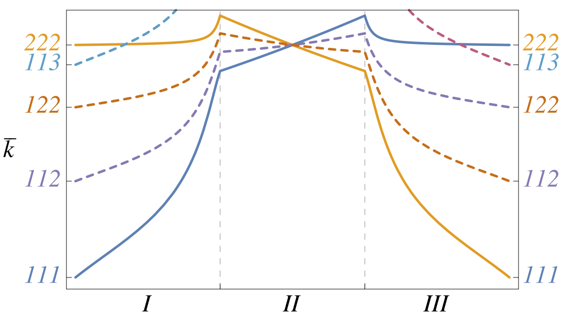

Figure 4. Parametric evolution of normalized wavenumbers for $N = 3$. Other parameters are the same as in Figure 3.

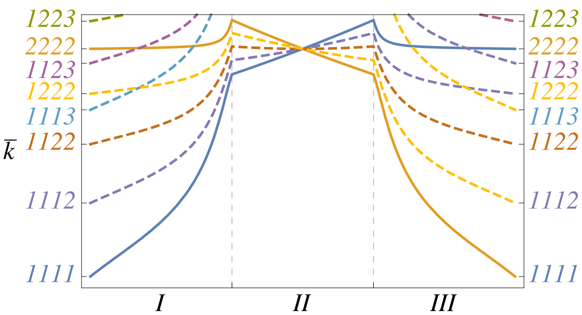

Figure 5. Parametric evolution of normalized wavenumbers for $N = 4$. Other parameters are the same as in Figure 3.

interaction $\langle 113|V|2^{\otimes 3}\rangle$, which vanishes since V is a two-body interaction, and the set of quantum numbers $(1, 1, 3)$ and $(2, 2, 2)$ has no common quantum number.

Still, there may be a tiny avoided crossing whose magnitude can be explained by the standard second-order perturbation theory. We may expect that the diabatic process induces the complete population inversion whose final state is $|2^{\otimes 3}\rangle$. Also, even if the interaction strength λ is moderately large, where the topology of the level diagram remains unchanged except that the avoided crossing becomes noticeable, the final state should be $|113\rangle$, whose energy is far larger than the ground state. In this sense, a incomplete population inversion should be realized.

C. The population inversion for $N > 2$

We shall prove that the adiabatic cycle C delivers $|1^{\otimes N}\rangle$ to $|2^{\otimes N}\rangle$ for $N > 2$, even in the presence of the two-body interparticle interaction. Here we explain the selection rule for arbitrary N (> 2), and examine each part of the cycle C. For example, $N = 4$ case is shown in Figure 5.

We explain the selection rule of the two-body interparticle interaction for $N > 2$. Namely, we examine the matrix element $\langle n'_1 n'_2 \dots n'_N|V|n_1 n_2 \dots n_N\rangle$. The matrix element vanishes when the two sets of quantum numbers $(n'_1, n'_2, \dots, n'_N)$ and $(n_1, n_2, \dots n_N)$ has, at least, three different elements, i.e., the number of the common quantum numbers is equal to $N - 3$ or less. In other words, non-vanishing matrix element has the following $\langle n'_1 n'_2 n_3 \dots n_N|V|n_1 n_2 n_3 \dots n_N\rangle$ where $(n_3, \dots, n_N)$ are the common quantum numbers.

We examine the first part C_I of C. We assume that the system is initially in the ground state $|1^{\otimes N}(\lambda)\rangle$. According to the selection rule, it is sufficient to examine $|1^{\otimes N-2}\psi, \phi\rangle$, where $|\psi\rangle$ and $|\phi\rangle$ are single particle adiabatic states, e.g. $|2\rangle$. Now we examine whether the eigenenergies of these states are degenerate. This is equivalent to compare the eigenenergies corresponding to $|11\rangle$ and $|\psi, \phi\rangle$ of the two particle system. As is seen in Figure 3, there is no level crossing in C_I. In this sense, there is no effective level crossing with the level $|1^{\otimes N}(\lambda)\rangle$, during C_I.

As for C_{III}, we conclude from an argument similar to the above, that the energy level corresponding to $|2^{\otimes N}(\lambda)\rangle$ has no effective level crossing.

Next we examine C_{II}, where the system is in $|R_1^{\otimes N}\rangle$. According to the selection rule, it suffices to examine $|R_1^{\otimes N-2}\psi, \phi\rangle$ with single particle adiabatic states $|\psi\rangle$ and $|\phi\rangle$. To clarify the level crossing, we compare $|R_1 R_1\rangle$ with $|\psi, \phi\rangle$. There are three cases. First, the levels corresponding to $|R_1 R_1\rangle$ and $|R_n, R_{n'}\rangle$ $((n, n') \neq (1, 1))$ do not occur. Second, the levels corresponding to $|R_1 R_1\rangle$ and $|R_n, L_{n'}\rangle$ exhibits a degeneracy only when $n = 1$ and $n' = 1$, where the corresponding matrix element involves a single-particle tunneling. Third, the levels corresponding to $|R_1 R_1\rangle$ and $|L_n, L_{n'}\rangle$ exhibits a degeneracy only when $n = 1$ and $n' = 1$, where the corresponding matrix element involves a two-particle tunneling. Since the matrix elements involving a tunneling contribution is exponentially small, the resultant gap should be also small. Hence, the diabatic process should occur even when the speed of the impenetrable wall is moderately slow.

V. Discussion and Summary

We argue here that the experimental realization of the population inversion suggested in this paper is feasible with the current state of the art. For example, we may utilize the scheme [22] to realize δ-wall with an approximate Gaussian wall.

Another possibility is to use a heavy particle as a wall, whose position may be manipulated by, say, an optical tweezer. The effective interaction between the wall particle and other particle may be tuned by external fields.

We note that the present scheme may offer a way to realize other exotic nonequilibrium states. Let us suppose, for example, the state of bosons is in $|2^{\otimes N}(\lambda)\rangle$, which can be generated from the adiabatic cycle C. After the interparticle interaction λ is adiabatically increased to ∞, the system arrives the higher excited state of the Tonks-Girardeau system, which may be described by the Lieb-Linigher model with the infinite interparticle interaction strength [23][24]. Similarly, after λ is adiabatically decreased to $-\infty$, the system now arrives at the higher excited state of the super-Tonks-Girardeau system [6]. This state is a much more highly-excited state compared to the super-Tonks-Girardeau state, because the initial state $|2^{\otimes N}(\lambda)\rangle$ is a higher excited state of noninteracting bosons.

In summary, we have shown that the adiabatic cycle C induces the nearly complete population inversion of the multi-boson system, when the interparticle interaction is not too strong. As pointed out in [18] for a single particle case, the present scheme may be extended to the case of an arbitrary shape of the confinement potential $V(x)$.

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