On the Singular Steady-State Output in Discrete-Time Linear Systems

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***Abstract*—The paper deals with discrete-time systems defined by difference equations whose transfer functions may have poles on the unit circle. Contrarily to the regular cases the eigenfunctions of these systems are no longer the exponentials. It is shown that if the input is a product of a falling factorial by an exponential the output is a linear combination of this kind of functions. In particular, the very useful and well-known ARIMA case is studied and exemplified.**

Keywords–ordinary difference equations; constant coefficient; particular solution; eigenfunction; transfer function; singular difference equation

I. INTRODUCTION

The constant coefficient ordinary difference equations have a long tradition in applied sciences and have a large amount of enginnering applications, mainly in Signal Processing [1], [3], [4], [10] where they are referred as ARMA (Autoregresive Moving Average) models. In these fields the difference equations are written in the general format

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{1}$$

wkere $n, M, N \in \mathbf{Z}$ and the coefficients $a_k,\ k = 0, 1, \ldots, N$ and $b_k,\ k = 0, 1, \ldots, M$ are real constants. Although we could consider fractional delays as in [5], we will not do it here.

In the regular case the response of these systems to a sinusoid is also a sinusoid with the same frequency [9], [10] which leads to introduce the frequency response that is another way of describing the system. In a general formulation we can say that the exponentials $\beta^n,\ n \in \mathbf{Z}\, \beta \in \mathbf{C}$ are the eigenfunctions of these systems.

The situation is not so simple in the singular case that we will study in this paper. However and as we will show the role of the exponentials is played by functions defined as the product of a falling factorial by an exponential. Let $(n)_k = n(n-1)(n-2)\ldots(n-k+1)$ be the Pochammer symbol for the falling factorial. We will assume that the input, $x(n)$, is the product of a falling factorial and an exponential defined on $\mathbf{Z}$:

$$x(n) = (n)_K \beta^n \tag{2}$$

where β is any complex number. This function does not have either Z transform or Fourier transform [10]. As we will show, these functions are not eigenfunctions, but we can state: when the input of the system is a function of the type (2) the output is a linear combination of several similar functions. This statement is valid for any regular or irregular system although we will pay a special attention to the irregular cases, mainly the Autoregresive Integrated Moving Average (ARIMA) models. So the frequency responses of these systems loose the normal interpretation. This problem was never considered with generality.

The procedure presented here is formally similar to the one followed in [6]–[8] for dealing with systems defined by differential equations.

We will start by introducing the eigenfunctions of difference equations and compute the corresponding eigenvalues. These are used to obtain the particular solutions we are looking for. Several examples are presented to illustrate the behaviour of the approach.

The objective of this paper is the study of singular cases corresponding to the situations where the transfer function becomes infinite; such situations are treated with all the generality. The important ARIMA model is a particular case with a pole at 1. We will show how to compute the output for these cases.

The paper outline is as follows. In section II, we will introduce the exponentials as eigenfunctions of the ARMA systems. The generalisation for the input as in (2) is done in section III. The singular cases are treated in section IV where the particular ARIMA. At last, we will present some conclusions.

II. THE EXPONENTIALS AS EIGENFUNCTIONS

The discrete convolution is defined by:

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k) y(n-k), \quad n \in \mathbf{Z} \tag{3}$$

This operation has several interesting properties, but we will study only those interesting for the development we intend to do.

1) Let the Kronecker delta be defined by

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \tag{4}$$

As it is easy to verify, this function is the neutral element of the convolution

$$x(n) = \delta(n) * x(n)$$

2) The convolution is commutative
In fact

$$x(n) * y(n) = y(n) * x(n)$$

as it is easily verified with the substitution $m = n-k$ in (3).

3) A shift in one factor produces the same shift in the convolution. Let $z(n) = x(n) * y(n)$. Then

$$x(n-n_0) * y(n) = z(n-n_0)$$

 ISBN: 978-1-61208-487-9

and using the commutativity

$$y(n-n_0)*x(n)=x(n-n_0)*y(n)$$

For proof we start from (3)

$$x(n-n_0)*y(n)=\sum_{k=-\infty}^{\infty}x(k-n_0)y(n-k)$$

and substitute m for $k-n_0$ to get

$$x(n-n_0)*y(n)=\sum_{m=-\infty}^{\infty}x(m)y(n-n_0-k)$$

With these properties at hand we return to our objective of computing the eigenfunction for equation (1).
Consider a particular input $x(n)=\delta(n)$ and let the corresponding solution be $h(n)$ that we will call Impulse Response. So, this is the solution of

$$\sum_{k=0}^{N}a_k h(n-k)=\sum_{k=0}^{M}b_k\delta(n-k) \tag{5}$$

Now convolve both sides in (5) with $x(n)$.

$$\sum_{k=0}^{N}a_k h(n-k)*x(n)=\sum_{k=0}^{M}b_k\delta(n-k)*x(n)$$

Using the above properties of the convolution we can write

$$\sum_{k=0}^{N}a_k\left[h(n-k)*x(n)\right]=\sum_{k=0}^{M}b_k x(n-k)$$

A comparison of this equation with (1) allows us to conclude that its solution is given by

$$y(n)=h(n)*x(n) \tag{6}$$

This means that the solution of (1) is the convolution of $x(n)$ with the impulse response.

Theorem 2.1: - The particular solution of the difference equation (1) when $x(n)=z^n,\ z\in\mathbf{C},\ n\in\mathbf{Z}$ is given by

$$y(n)=H(z)z^n \tag{7}$$

provided that $H(z)$ exists.

This theorem shows that the exponentials are the eigenfunctions of the constant coefficient ordinary difference equations.

Proof: Insert $x(n)=z^n$ into (6) and use (3) to get

$$y(n)=\sum_{k=-\infty}^{\infty}h(k)z^{n-k}=\sum_{k=-\infty}^{\infty}h(k)z^{-k}z^n$$

with

$$H(z)=\sum_{k=-\infty}^{\infty}h(k)z^{-k} \tag{8}$$

we obtain (7). $H(z)$ is called *Transfer Function* of the system defined by the difference equation (1) and is the Z transform of the impulse response. ■

Inserting (7) into (1) we conclude immediately that

$$H(z)=\frac{B(z)}{A(z)}=\frac{\sum\limits_{k=0}^{M}b_k z^{-k}}{\sum\limits_{k=0}^{N}a_k z^{-k}} \tag{9}$$

In the following we will consider that the *characteristic polynomial* in the denominator is not zero for the particular value of z at hand. Later we will consider the cases where the characteristic polynomial is zero (z is a pole).

Example 1

Let $x(n)=2^n$ and consider the equation

$$y(n)=x(n)+x(n-1)$$

We have $H(z)=1+z^{-1}$. So the particular solution is given by $y(n)=H(2)2^n=\frac{3}{2}.2^n$ Let now $x(n)=(-1)^n$. We have $y(n)=H(-1)(-1)^n\equiv 0$

Example 2

Consider the difference equation

$$y(n)+y(n-1)-4y(n-2)+2y(n-3)=x(n)+2x(n-1)$$

Let $x(n)=(1/2)^n$. The solution is given by:

$$y(n)=\frac{1+2(1/2)^{-1}}{1+(1/2)^{-1}-4(1/2)^{-2}+2(1/2)^{-3}}(1/2)^n=\frac{5}{3}(1/2)^n$$

The sinusoidal case: – In a particular setting, put $z=e^{i\omega_0}$. We obain immediately

$$y(n)=H(e^{i\omega_0})e^{i\omega_0 n}$$

Example 3

Consider the difference equation

$$y(n)+y(n-1)-4y(n-2)+y(n-3)=x(n)$$

Let $x(n)=e^{i\frac{\pi}{2}n}$. The solution is given by:

$$y(n)=\frac{1}{1+i^{-1}-4i^{-2}+i^{-3}}e^{i\frac{\pi}{2}n}=\frac{1}{5}e^{i\frac{\pi}{2}n}$$

This is very interesting since it allows us to compute easily the solution when $x(n)=\cos(\omega_o t)$ or $x(n)=\sin(\omega_o t)$. Consider the first case; the second is similar. We have

$$x(n)=\cos(\omega_o t)=\frac{1}{2}e^{i\omega_0 n}+\frac{1}{2}e^{-i\omega_0 n}$$

that leads to

$$y(n)=H(e^{i\omega_0})\frac{1}{2}e^{i\omega_0 n}+H(e^{-i\omega_0})\frac{1}{2}e^{-i\omega_0 n}$$

The function $H(e^{i\omega})=\left|H(e^{i\omega})\right|e^{i\varphi(e^{i\omega})}$ is called *frequency response* in engineering applications. The function $\left|H(e^{i\omega})\right|$ is the *amplitude spectrum* and is an even function, while $\varphi(e^{i\omega})$ is the *phase spectrum* and is an odd function, if the coefficients in (1) are real.

Theorem 2.2: - The particular solution of the difference equation (1) when $x(n)=\cos(\omega_o n)$ is given by

$$y(n)=\left|H(e^{i\omega_0})\right|\cos\left[\omega_o n+\varphi(e^{i\omega_0})\right] \tag{10}$$

Proof: According to what we said above, $\left|H(e^{-i\omega})\right| = \left|H(e^{i\omega})\right|$ and $\varphi(e^{-i\omega}) = -\varphi(e^{i\omega})$ which leads to

$$y(n) = \left|H(e^{i\omega_0})\right| \frac{1}{2}\left[e^{i\omega_0 n}e^{i\varphi(e^{i\omega})} + e^{-i\omega_0 n}e^{-i\varphi(e^{i\omega})}\right]$$

that leads immediately to the result. ■

It is important to remark that when $H(e^{i\omega_0}) = 0$, $y(n)$ is identically null. This is the reason why we call *filters* the systems described by linear difference equations. This theorem states clearly the importance of the frequency response of a system.

Example 4

Consider again the above equation, but change the second member:

$$y(n) + y(n-1) - 4y(n-2) + y(n-3) = 3x(n) - 4x(n-1)$$

and assume that $x(n) = \sin\left(\frac{\pi}{2}n\right)$. Then

$$H(z) = \frac{3 - 4z^{-1}}{1 + z^{-1} - 4e^{-2} + z^{-3}}$$

and

$$y(n) = \frac{1}{2i}\frac{3 - 4e^{-i\pi/2}}{1 + e^{-i\pi/2} - 4e^{-i\pi} + e^{-i3\pi/2}}e^{i\frac{\pi}{2}n}$$
$$-\frac{1}{2i}\frac{3 - 4e^{i\pi/2}}{1 + e^{i\pi/2} - 4e^{i\pi} + e^{3\pi/2}}e^{-i\frac{\pi}{2}n}$$

leading to

$$y(n) = \sin\left(\frac{\pi}{2}n + \varphi\right)$$

with $\varphi = \arctan(4/3)$

III. Functions equal to the product of a falling factorial by an exponential

To go further we are going to consider the case $x(n) = (n)_K\beta^n, \quad n \in \mathbf{Z}, K \in \mathbf{N}_0$. Although not so important as the previous case, it constitutes a simple generalization that is interesting from analytical point of view. It is not difficult to see that we can write $x(n) = \beta^K \lim\limits_{z\to\beta} \frac{d^K}{dz^K}z^n$. Return to (6) and particularise for our case to obtain:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)(n-k)_K z^{n-k} = \sum_{k=-\infty}^{\infty} h(k)\frac{d^K}{dz^K}z^{n-k}$$

For z in the region of convergence of the Z transform the series converges uniformly and we can commute the derivative and summation operations. This procedure leads to the next theorem.

Theorem 3.1: - The particular solution of the difference equation (1) when $x(n) = (n)_K\beta^n$ is given by

$$y(n) = \beta^K \lim_{z\to\beta}\frac{d^K\left[H(z)z^n\right]}{dz^K} \tag{11}$$

Using the Leibniz rule we can obtain another expression for $y(n)$ stated in as follows.

Theorem 3.2: - The particular solution of the difference equation (1) when $x(n) = (n)_K\beta^n$ is given by:

$$y(n) = \sum_{j=0}^{K}\binom{K}{j}H^{(j)}(\beta)(n)_{K-j}\beta^n \tag{12}$$

provided that β is not a pole of the transfer function.

In particular, when $x(n) = (n)_K$ the solution is given by:

$$y(n) = \sum_{j=0}^{K}\binom{K}{j}H^{(j)}(1)(n)_{K-j} \tag{13}$$

For $K = 0$, $x(n) = 1$ and $y(n) = H(1)$.

Example 5

Return back to the above example $y(n)+y(n-1)-4y(n-2)+y(n-3) = 3x(n)-4x(n-1)$ and put $x(n) = n$. We obtain immediately

$$y(n) = \sum_{j=0}^{1}\binom{1}{j}H^{(j)}(1)(n)_{1-j}$$

As $H(z) = \frac{3-4z^{-1}}{1+z^{-1}-4e^{-2}+z^{-3}}$, $H(1) = 1$ and $H'(1) = 0$ the solution is

$$y(n) = n$$

IV. The singular case - ARIMA

Consider now the situation where the *characteristic polynomial* in the denominator has an m^{th} order root for $z = \beta$. To look for a solution assume that $x(n) = w(n)\beta^n$ and

$$y(n) = v(n).\beta^n \tag{14}$$

Insert $x(n)$ and $y(n)$ into (1) to obtain a new equation

$$\sum_{k=0}^{N} a_k\beta^{-k}v(n-k) = \sum_{k=0}^{M} b_k\beta^{-k}w(n-k) \tag{15}$$

with transfer function $H(\beta z)$. In fact we moved the root of $A(z)$ from $z = \beta$ to $z = 1$. This means that we have a m^{th} order pole at $z = 1$. We can say that we transformed the singular system into an ARIMA system that appears frequently in enonometric studies. In terms of the variable n we have a m^{th} order differentiation at the output. This is equivalent to do an anti-difference on the input. Now perform a new substitution $u(n) = D^m v(n)$ where D means the differencing operation $Dv(n) = v(n) - v(n-1)$ to obtain

$$\sum_{k=0}^{N-m}\bar{a}_k u(n-k) = \sum_{k=0}^{M}\bar{b}_k w(n-k) \tag{16}$$

where $\bar{a}_k,\ k = 0, 1, \ldots, N-m$ are the coefficients of the new characteristic polynomial $\bar{A}(z) = \frac{A(\beta z)}{(1-z^{-1})^m}$ and numerator polynomial $\bar{B}(z) = B(\beta z)$.

For the particular case we are interested in, $w(n) = (n)_K$ we can use (13). Let D^{-1} represent the anti-difference – $D^{-1}Df(n) = DD^{-1}f(n) = f(n)$ – essentially the m^{th} order primitive without primitivation constants. So $v(n) = D^{-m}u(n)$, allowing to obtain the following result.

Theorem 4.1: - The particular solution of the difference equation (1) when $x(n) = (n)_K \beta^n$ with $A(\beta) = 0$ is given by

$$y(n) = \beta^n D^{-m} \left[\sum_{j=0}^{K} \binom{K}{j} \bar{H}^{(j)}(1)(n)_{K-j} \right] \quad (17)$$

with

$$\bar{H}(z) = \frac{\bar{B}(z)}{\bar{A}(z)} = \frac{(1-z^{-1})^m B(\beta z)}{A(\beta z)}$$

It is not difficult to show that (17) can be written as

$$y(n) = \beta^n \left[\sum_{j=0}^{K} \binom{K}{j} \bar{H}^{(j)}(1) \frac{(K-j)!}{(K+m-j)!} (n)_{K+m-j} \right] \quad (18)$$

where we used the following recursively obtained result

$$D^{-m}(n)_K = \frac{K!}{(K+m)!}(n)_{K+m} \quad (19)$$

If $K = 0$ (pure exponential input), we obtain:

$$y(n) = \beta^n \bar{H}(1) \frac{1}{m!}(n)_m \quad (20)$$

If we make $\beta = e^{i\omega_0 n}$ we are led to conclude that the response of the ARIMA model to a pure sinusoid is never a pure sinusoid: the amplitude increases with time. This is the reason why this model is used for modeling non-stationary situations.

Example 6

Consider the following equation with $x(n) = n(-1)^n$

$$y(n) - y(n-1) - 4y(n-2) - 2y(n-3) = x(n)$$

The point $z = -1$ is a pole of the transfer function, $A(-1) = 0$, of order $m = 1$. On the other hand, $\bar{H}(z) = \frac{1+z^{-1}}{1-z^{-1}-4z^{-2}-2z^{-3}} = \frac{1}{1-2z^{-1}-2z^{-2}}$ and $\bar{H}'(z) = -\frac{-2z^{-2}-4z^{-3}}{(1-2z^{-1}-2z^{-2})^2}$, leading to $\bar{H}(-1) = 1$ and $\bar{H}'(-1) = 2$. The solution is $y(n) = [1/2(n)_2 + 2n](-1)^n$.

Example 7

Consider the following ARIMA equation with $x(n) = 1$

$$y(n) - 2y(n-1) + 3y(n-2) - 2y(n-3) = x(n)$$

The point $z = 1$ is a pole of the transfer function, $A(1) = 0$, of order $m = 1$. On the other hand,

$$\bar{H}(z) = \frac{1}{1 - z^{-1} + 2z^{-2}}$$

leading to $\bar{H}(1) = 1/2$. The solution is $y(n) = n/2$.

Example 8

The oscillator is a very interesting system that can be defined by the equation

$$y(n) - 2\cos(\omega_0)y(n-1) + y(n-2) = x(n) - \cos(\omega_0)x(n-1)$$

Now, let $x(n) = e^{i\omega_0 n}$. The system has two simple ($m = 1$) poles at $e^{\pm i\omega_0 n}$. So, $\bar{H}(e^{i\omega_0}) = 1/2$ and the output is easily obtained

$$y(n) = \frac{1}{2} n e^{i\omega_0 n}$$

As we said above, it is a non-stationary model.

V. Conclusions

The singular steady-state output in discrete-time linear systems was studied using an eigenfunction approach to the computation of the steady-state output. Products of falling factorial and exponentials were used as inputs and the corresponding outputs computed in a simple way. Some examples were used to illustrate the procedure, in particular the ARIMA case was considered.

This formulation can be used to study the autocorrelation function of the output when the input is a stationary stochastic process.

Acknowledgments

This work was partially supported by National Funds through the Foundation for Science and Technology of Portugal under project PEst–UID/EEA/00066/2013

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