Reachability Games Revisited

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***Abstract*— In this paper, we provide a refined analysis of the classical algorithm for solving reachability games. We provide a new algorithm that remembers information about fewer nodes than the classical algorithm does by computing the number of *efforts* made by the player to win the game.**

Keywords – reachability games; effort based strategies; memoryless determinacy

I. INTRODUCTION

In system Verification and Testing, the *reachability* question asking if a system can attain some specified state from a given state is well studied and well motivated. The reachability question was studied in the context of games by many [1] – [7]. Reachability games are played between two players Players 0 and Player 1, over a finite directed graph. The nodes of the graph are the states of the system it models and edges of the graph represent transitions of the system. An infinite sequence of states of the system can now be viewed as an infinite path through the graph. The question of reachability in verification can now be solved in terms of constructing winning strategies for the corresponding reachability game. In reachability games, Player 0 wins a play if the play visits some specified set, called a target set, at least once. A reachability game is solved in linear time on the size of the underlying graph [2].

The problem of solving a reachability game is mainly about constructing the *attractor* set for the winner. The concept of attractor set is also useful for the solution of infinite games with Safety [6], Buchi [8], McNaughton [9], and, Parity [10] winning conditions. The classical algorithm (see for instance [2]) for solving reachability games suggests a winning strategy that remembers ranks of all the nodes in the winning region of Player 0. Roughly, a rank of a node v is i if Player 0 can reach the target set (starting from v) within i moves made by the players. In this paper, we carefully analyse the attractor set of the target set. We call the attractor set (for Player 0) of the target set the winning region for Player 0. We observed that the winning region may contain Player 0 nodes such that all its outgoing edges lead to the winning region of Player 0. We call a set containing such nodes the *effortless region* of Player 0. The ranks of the nodes in effortless region need not to be remembered. Therefore, our winning strategy for Player 0 takes into account such nodes and hence improves upon memory efficiency. We call such strategies, *effort-based* strategies. We call a strategy that is dependent only on the current node of the play, *memoryless* strategy. Thus memoryless strategies do not depend on the *history*, where history is a finite prefix of a play. We will prove that effort-based strategies are memoryless strategies. Such strategies recall fewer ranks than the classical approach.

The summary of the paper is as follows. In Section II, we will provide basic definitions about games in general. In Section III, we will describe reachability games. In Section IV, we will provide our own procedure for solving reachability games. Finally, the conclusion is presented in Section V.

II. BASIC DEFINITIONS

Our games are played between two players. We call them Player 0 and Player 1. The underlying graph of a game is called arena. Our definition of arena is the following:

Definition 2.1 (Arena): An *arena* is a tuple $(V_0 \cup V_1, E)$, where V_0 and V_1 are pairwise disjoint sets of nodes and $E \subseteq V_0 \times V_1 \cup V_1 \times V_0$ is the set of edges. We set $V = V_0 \cup V_1$. We also postulate that for every $u \in V$ there always exists $v \in V$ such that $(u, v) \in E$. We always assume that the set V of nodes is finite. The set of *successors* of $u \in V$ is defined by $uE = \{v \in V \mid (u, v) \in E\}$.

Thus, arenas are just finite bipartite graphs. The nodes of the set V_0 will be called Player 0's nodes and nodes of V_1 will be called Player 1's nodes.

Let $G = (V_0 \cup V_1, E)$ be an arena. A play between Player 0 and Player 1 is described as follows. The play begins at any node $v_0 \in V$. Say the node is in V_0. In this case, Player 0 selects an edge $e = (v_0, v_1)$, moves along the edge, and passes control to the opponent. Then, Player 1 selects an edge $e = (v_1, v_2)$, moves along the edge and passes control to the opponent. This goes on turn by turn. The definition of arena implies that at any given moment of the play, the players are able to make moves and continue the play. Formally, we define a play as follows.

Definition 2.2 (Play): A *play* in the arena $G = (V_0 \cup V_1, E)$ is an infinite sequence $v_0,\ v_1,\ v_2,\ v_3\ \ldots$ such that (v_0, v_1), (v_1, v_2), (v_2, v_3), $\ldots$ are all edges of the graph.

Unless the arena is trivial, there are infinitely many infinite plays. Clearly, every play is an element of V^ω, where V^ω is the set containing all sequences over V.

Let $G = (V_0 \cup V_1, E)$ be an arena. Let $\rho = v_0, v_1, v_2, \ldots,$ be an infinite play played between Player 0 and Player 1. We would like to define what it means that Player 0 wins the play ρ. The winner is determined through the following definition.

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Definition 2.3: A *winning set* for Player 0 is a subset $W \subseteq V^\omega$. We say that Player 0 *wins* the play ρ if $\rho \in W$. Otherwise, Player 1 wins the play.

Now, we are ready to formally define a game.

Definition 2.4 (Game): A game between Player 0 and Player 1 is a tuple $(V_0 \cup V_1, E, W)$, where $G = (V_0 \cup V_1, E)$ is an arena and W is a winning set for Player 0. We typically denote games by Γ.

Note that, arenas are finite objects while winning sets W are not necessarily objects that are defined by finite means.

A. Strategies

In this section, given a game, we define the winner of the game and explain what it means to solve the game. The notation $\sigma \in \{0,\ 1\}$ will represent one of the two players Player 0 and Player 1, his opponent will be represented by $1 - \sigma$.

Let $\Gamma = (V_0 \cup V_1, E, W)$ be a game. We define histories of the game as follows.

Definition 2.5: A *history* is a finite prefix of a play. We define set of histories for Player σ as follows: $H(\sigma) = \{h \mid h$ is a history and the last letter of h is in $V_\sigma\}$.

Clearly, $H(\sigma) \cap H(1-\sigma) = \emptyset$ and every history is either in $H(\sigma)$ or in $H(1-\sigma)$. Informally, a strategy for Player σ, is a rule that tells the player which edge to select given a history of a play in $H(\sigma)$. Formally, we define a strategy for a player as follows.

Definition 2.6: A *strategy* for Player σ is a function $f_\sigma : H(\sigma) \to V$ such that for every $h = v_0, v_1, \ldots, v_n \in H(\sigma)$ we have $(v_n, f_\sigma(h)) \in E$.

Now, given a node $v \in V$ and strategy f_σ for Player σ, one can consider all the plays starting at v and consistent with the strategy f_σ. Here, we say that a play $\rho = v_0, v_1, v_2, \ldots$ is *consistent with* f_σ if $v_0 = v$ and for all histories $h = v_0, v_1, \ldots v_i \in H(\sigma)$ of this play we have $(v_i,\ f_\sigma(h)) \in E$ and $v_{i+1} = f_\sigma(h)$.

Definition 2.7: Let $\Gamma = (V_0 \cup V_1, E, W)$ be a game. Let $v \in V$ be a node.

- We say *Player σ wins from a node v* if Player σ has a winning strategy f_σ such that all the plays that begin from v and consistent with f_σ are winning for the player.
- We say *Player σ wins from a set* or has a *winning strategy from a set $A \subseteq V$* if Player σ has a winning strategy from each node in A.
- We say that v is a *winning node for* Player σ if the player wins the game from the node v.

From this definition, it follows that, if v is a winning node for a player, then v can not be a winning node for the opponent. One of the fundamental concepts in game theory is the following definition.

Definition 2.8: A game Γ is *determined* if every node of the game is winning for either Player 0 or Player 1.

There are examples of games that are not determined, see [11]. Determinacy is one of the important topics in descriptive set theory. One of the important theorems is the following theorem of Martin [12].

Theorem 2.9 (Martin's determinacy theorem): Every Borel game, that is the game at which W is a Borel set, is determined.

Reachability games are Borel and hence determined. The definition below is meant when we say a game is solved.

Definition 2.10: Let Γ be a game. We say Γ is solved if there exists an algorithm that given the Γ, outputs the sets W_0 and W_1, where W_0 is the set of all nodes in Γ from which Player 0 wins the game and W_1 is the set of all nodes in Γ from which Player 1 wins the game. The set W_σ, $\sigma \in \{0, 1\}$ is called winning region for Player σ.

III. Reachability Games

In this section, we discuss reachability games in detail. The algorithm and definitions discussed here are borrowed from [2]. In reachability games, Player 0 wins a play if the play visits a specified set of nodes at least once. Formally, we define reachability games as follows.

*Definition 3.1 (*Reachability Games): A *reachability game* Γ consists of:

1) The arena $G = (V_0 \cup V_1, E)$.
2) The target set T of nodes $T \subseteq V_0 \cup V_1$.

We say that *Player 0 wins a play* $v_0,\ v_1,\ v_2,\ v_3\ \ldots$ if there exists an i such that $v_i \in T$. Otherwise, Player 1 wins the play.

From the definition, it is clear that Player 0 wins a play ρ from a node u if

- ρ begins from the node u;
- there is a finite prefix η of ρ such that the last node in η belongs to the target set.

Definition 3.2: A *memoryless strategy* for Player σ is a function $f_\sigma : V_\sigma \to V$ such that $(u, f_\sigma(u)) \in E$. A game enjoys *memoryless determinacy* if for every node one of the players wins the game with memoryless strategy.

It turns out that winners in reachability games have memoryless winning strategies. We prove this in the next theorem. Before we proceed, we define some notations. Let Γ be a reachability game. Assume $X \subseteq V$. Define,

$$\mathsf{reach}_\sigma(X) = \{u \in V_\sigma \mid \exists\ v \in uE \cap X\} \cup \\ \{u \in V_{1-\sigma} \mid uE \subseteq X\}.$$

When the player is clear, then sometimes we denote the above set by $\mathsf{reach}(X)$.

Theorem 3.3 (Memoryless Determinacy [2] pp. 34): Reachability games enjoy memoryless determinacy.

Proof: The winning region for Player 0 is defined inductively. We set, $X_0 = T$, and for $i \in \omega$, $X_{i+1} = \mathsf{reach}_0(X_i) \cup X_i$. Since the set of nodes V is finite there is an s such that $X_s = X_{s+1}$, where s is the smallest such number.

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Claim 3.4: The set X_s is the winning region for Player 0, that is $W_0 = X_s$.

For the proof, we use the concept of rank. We say a node u has *rank* r, $r \geq 0$, if $u \in X_r \setminus X_{r-1}$. A node u has infinite rank if $u \notin X_r$ for all r.

To define a memoryless strategy f_0 for Player 0 we linearly order $<$ the set V_1 that is, $v_1 < v_2 < v_3 < \cdots < v_l$ where $l = |V_1|$. We define f_0 as follows:

Let $v \in X_s \cap V_0$. Let r be the rank of v. Then $f_0(v)$ is minimal with respect to $<$ such that $(v, f_0(v)) \in E$ and rank of $f_0(v)$ is $r - 1$.
For $v \notin X_s$, we set $f_0(v)$ be the minimal with respect to the order $<$ such that $(v, f_0(v)) \in E$.

We show that f_0 is winning strategy from the set X_s. Let v be a node in X_s. From the above definition of $\mathsf{reach}(Y)$ it implies that v has some finite rank r say. If v is Player 0's node then by the strategy f_0 Player 0 chooses $f_0(v)$ of rank $r - 1$. If v is Player 1's node then Player 0 waits for Player 1's move. Since $vE \subseteq X_{r-1}$, any choice of Player 1 selects a node in vE of rank strictly less than r. Each player's move select a node of lesser rank every time. Since r is finite, every play that begins from v ultimately ends at a target node. Hence $X_s \subseteq W_0$.

Now, we show that Player 0 cannot win from a node in $V \setminus X_s$. Let $M = V \setminus X_s$. To define a memoryless strategy f_1 for Player 1 we linearly order the set V_0 that is $v'_1 < v'_2 < v'_3 < \cdots < v'_m$ where $m = |V_0|$.

If $v \in M \cap V_1$ then $f_1(v)$ is minimal with respect to the order such that $f_1(v) \in M$ and $(v, f_1(v)) \in E$.
If $v \in X_s$ then $f_1(v)$ is the minimal with respect to the order such that $(v, f_1(v)) \in E$.

Let v belong to M. If v is Player 0's node then Player 1 does nothing but just waits for the Player 0's move. Any choice of Player 0 selects a node in M. This is because $vE \subseteq M$ as otherwise v would be in W_0.

If v is Player 1's node then there exists a node $v' \in M$ such that $(v, v') \in E$ otherwise v would belong to W_0. By strategy f_1 Player 1 chooses minimal $f_1(v)$ with respect to the order $<$ such that $f_1(v) \in M$ and $(v, f_1(v)) \in E$. Player 0 cannot win any play which begins from a node belongs to M if Player 1 follows the strategy f_1. This is because $M \cap T = \emptyset$. Hence, $W_0 = X_s$ and $W_1 = M$. ■

Corollary 3.5: There exists an algorithm that solves reachability games in $O(|V| + |E|)$ time, where V is the set of nodes and E is the set of edges in the arena.

Proof: We construct the winning region for Player 0 inductively. Initially, we set $X_0 = T$. Suppose X_i is constructed. To construct X_{i+1}, first we copy elements of X_i to X_{i+1}. Second, we add a node u in X_{i+1} if:

- $u \in V_0$ and if there exists a node $v \in X_i$ such that $(u, v) \in E$, then we add to X_{i+1}.
- $u \in V_1$ and if $uE \subseteq X_i$, then we add to X_{i+1}.

To implement the procedure for constructing the winning region in $O(|V| + |E|)$ time, we assign a counter $c(u)$ to a node $u \notin T$. Initially, we set $c(u) = 1$ if $u \in V_0$ and $c(u) = |uE|$ if $u \in V_1$.

Whenever, we add a node v to X_{i+1}, where $v \notin X_i$, we subtract 1 from the counter of each node u such that $(u, v) \in E$; From this point on the edge (u, v) will never be used again. When a counter becomes zero then the node is also added to X_{i+1}. This shows that the running time of the algorithm is in $O(|V| + |E|)$. ■

Corollary 3.6: Let Γ be a reachability game. The function $\phi_\sigma : 2^V \to 2^V$ defined by $\phi_\sigma(A) = A \cup \mathsf{reach}_\sigma(A)$, is monotone function with respect to set inclusion, where $A \subseteq V$ and 2^V is the set containing all subsets of V. That is, $\phi_\sigma(A) \subseteq \phi_\sigma(B)$ whenever $A \subseteq B$.

Definition 3.7: We denote the winning region of Player 0 in a reachability game by $\mathsf{Attr}_0(T)$ and call it the *0-attractor* of the set T. A memoryless winning strategy f_0 as described in the proof of Theorem 3.3 is called *T-attractor strategy* for Player 0. When the target set T is clear, then we simply say attractor strategy for Player 0.

Note that, in reachability games, we can change the roles of the players. In this case, Player 1 tries to reach a given set T while the opponent tries to avoid it. As shown above, one can build the 1-attractor of the set T. Hence, we can talk about σ-attractor sets for the players when a target set is specified.

Definition 3.8: A *σ-trap* (or trap for σ) is a subset $X \subseteq V$ such that $vE \subseteq X$ for every $v \in X \cap V_\sigma$ and $vE \cap X \neq \emptyset$ for every $v \in X \cap V_{1-\sigma}$. A memoryless strategy which assigns for $v \in X \cap V_{1-\sigma}$ a node $f(v) \in vE \cap X$ is called a *trapping strategy* for Player $1 - \sigma$.

Corollary 3.9: The complement of σ-attractor of a target set T is σ-trap.

IV. Reachability games and effort moves

In this section, We give a refined analysis of the set $\mathsf{Attr}_0(T)$ for a given reachability game Γ. The idea here is to compute the number of efforts made by Player 0 to win the game from $\mathsf{Attr}_0(T)$. Let $X \subseteq V$ be a set. Define $\Im(X) = \{v \in V \mid vE \subseteq X\}$. We define the sequence $\Im_0, \Im_1, \Im_2, \Im_3, \ldots$ as follows:

$$\Im_0 = X, \quad \Im_{i+1} = \Im(\Im_i) \cup \Im_i.$$

Since the arena is finite, there exists a minimal k such that $\Im_{k+1} = \Im_k$. We call this $\Im_k$, the *effortless region for X* and denote it by $\mathsf{eff}(X)$.

Lemma 4.1: Player 0 has a winning strategy from $\mathsf{eff}(T)$ to visit T.

Proof: Let $u \in \mathsf{eff}(T)$. This implies that $u \in \Im_i$ for some i. Since $uE_i \subseteq \Im_{i-1}$, any play starting at u will eventually visit $\Im_0 = T$. Thus, a winning strategy for Player 0 is simply to choose any node v such that $(u, v) \in E$. ■

In order to construct the winning region for Player 0, we define the sequence eff_0, eff_1, eff_2, $\mathsf{eff}_3, \ldots$ as follows:

$$\mathsf{eff}_0 = \mathsf{eff}(T)$$

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$$\mathsf{eff}_{i+1} = \mathsf{eff}(\mathsf{eff}_i \cup \mathsf{reach}(\mathsf{eff}_i)).$$

Let us recall $\mathsf{reach}(Y) = \{u \in V_0 \mid \exists v \in uE \cap Y\} \cup \{u \in V_1 \mid uE \subseteq Y\}$, where $Y \subseteq V$. Here, the second part is empty. Since, the arena is finite, it implies that there exists a minimal t such that $\mathsf{eff}_{t+1} = \mathsf{eff}_t$.

Definition 4.2: The Player 0's move from a node u to a node v in a reachability game is called an *effort move* if $u \in \mathsf{reach}_0(\mathsf{eff}_i) \setminus \mathsf{eff}_i$ and $v \in \mathsf{eff}_i$ for some i.

Lemma 4.3: Player 0 has a strategy from $\mathsf{eff}_{i+1} \setminus \mathsf{eff}_i$ to visit T after $i+1$ effort moves have been made.

Proof: We prove the lemma by induction on i. For $i = 0$, let $x \in \mathsf{eff}_1 \setminus \mathsf{eff}_0$. This implies that any play from x will eventually visit $\mathsf{reach}_0(\mathsf{eff}(T))$. For every $u \in \mathsf{reach}_0(\mathsf{eff}(T))$, there exists a $v \in \mathsf{eff}(T)$ such that $(u, v) \in E$ otherwise $u \notin \mathsf{reach}_0(\mathsf{eff}(T))$. We set $f(u) = v$. Player 0 now moves to this v.

Let the lemma be true for $i = k$. Let $x \in \mathsf{eff}_{k+1} \setminus \mathsf{eff}_k$. Any play from x will eventually visit $\mathsf{reach}(\mathsf{eff}_k)$. For every $u \in \mathsf{reach}_0(\mathsf{eff}_k)$ there exists a $v \in \mathsf{eff}_k$ such that $(u, v) \in E$, otherwise $u \notin \mathsf{reach}_0(\mathsf{eff}_k)$. We set $f(u) = v$. If Player 0 follows this strategy f, then Player 0 can visit eff_k after one effort has been made if a game begins from the node x. By induction hypothesis Player 0 can visit T after $k+1$ effort moves. ■

Theorem 4.4: Let Γ be a reachability game. Consider the sequence defined as follows:

$$\mathsf{eff}_0 = \mathsf{eff}(T)$$

$$\mathsf{eff}_{i+1} = \mathsf{eff}(\mathsf{eff}_i \cup \mathsf{reach}_0(\mathsf{eff}_i)).$$

If t is the minimal number such that $\mathsf{eff}_{t+1} = \mathsf{eff}_t$ then eff_t is the winning region for Player 0 and $V \setminus \mathsf{eff}_t$ is the winning region for Player 1.

Proof: Let $u \in \mathsf{eff}_t$. By the above two lemmas Player 0 has a strategy to visit T after t effort moves. Hence $\mathsf{eff}_t \subseteq W_0$. This strategy can be written explicitly as follows. For any given $u \in V_0$ set:

> If $u \in \mathsf{reach}_0(\mathsf{eff}_i) \setminus \mathsf{eff}_i$ for some i then select v such that $(u, v) \in E$ and $v \in \mathsf{eff}_i$. Otherwise, select any w such that $(u, w) \in E$.

To prove $W_0 = \mathsf{eff}_t$, we show that Player 1 has a winning strategy from $V \setminus \mathsf{eff}_t$. We set $W' = V \setminus \mathsf{eff}_t$. Let a play begins from a node u in W'. If u is Player 0's node then Player 1 does nothing but just waits for the Player 0's move at u. Any choice of Player 0 selects a node in W'. This is because $uE \subseteq W'$ as otherwise u would be in $\mathsf{reach}_0(\mathsf{eff}_t)$. If $u \in V_1 \cap W'$, then there exists a node $v \in W'$ such that $(u, v) \in E$ otherwise u would belong to eff_t. We define $g(u) = v$. Any play which begins from a node in W' and consistent with this strategy g always stays inside W'. Hence g is a winning strategy for Player 1 from W' because $W' \cap T = \emptyset$. Thus, $W_0 = \mathsf{eff}_t$ and $V \setminus \mathsf{eff}_t = W_1$. ■

Note that the winning strategy f, for Player 0, extracted from the classical algorithm that solves a reachability game, remembers ranks of all the nodes in $\mathsf{Attr}_0(T)$. That is, for all $u \in \mathsf{Attr}_0(T)$, $f(u) = v$, where $v \in uE$ and rank of v is strictly less than u. For all $z \in V_0 \setminus \mathsf{Attr}_0(T)$, $f(z)$ is such that $(z, f(z)) \in E$. Our new procedure for solving reachability games suggests a winning strategy g that remembers only nodes in

$$T \cup \mathsf{reach}_0(\mathsf{eff}_0) \cup \mathsf{reach}_0(\mathsf{eff}_1) \cup \mathsf{reach}_0(\mathsf{eff}_2) \cup \ldots.$$

We define g as follows. For all $u \in X$, $g(u) = f(u)$, where $X = T \cup \mathsf{reach}_0(\mathsf{eff}_0) \cup \mathsf{reach}_0(\mathsf{eff}_1) \cup \mathsf{reach}_0(\mathsf{eff}_2) \cup \ldots$. For all $z \in V_0 \setminus X$, $f(z)$ is such that $(z, f(z)) \in E$. We call this strategy, *effort-based* strategy . Thus, we obtain the following theorem.

Theorem 4.5: Given a reachability game, there exists a linear time algorithm that extracts an effort-based memoryless winning strategy for the winner.

V. Conclusion

To win a reachability game, the classical algorithm remembers ranks of all nodes of the arena. The winning region for a player may contain nodes such that all their outgoing edges lead to the winning region of the player and hence no effort is involved by the player at such nodes. The region that contains such nodes we called it effortless region. Our algorithm takes effortless region into account and hence improved memory efficiency. Moreover, the algorithm is memoryless and it takes linear time on the number of nodes and edges of the arena.

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